Certain Theorems Concerning the Limits of Sequences of Continuous Functions

By Norbert Wiener

It is a familiar fact that the necessary and sufficient condition that a set of n-partite numbers have a limit is that it be infinite and bounded. Fréchet*

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has extended his theorem to points in a space of a denumerable infinity of dimensions; his proof, it is true, is incomplete in that it involves a selection from an infinitude of classes, and consequently depends on Zermelo's multiplicative axiom, but this difficulty can be readily removed. When, however, we make the transition from space of a denumerable infinity of dimensions to function-space, a bounded set need no longer have a limit in the sense of Fréchet — i.e., a limit towards which a selected non-repeating sequence converges uniformly.

The purpose of this paper is to develop the properties of an extended sort of boundedness, which we shall call constraint. We shall show that every constrained infinite set ~~and~~ of continuous functions contains at least one non-repeating sequence of functions which converges uniformly in a certain interval to a continuous limiting satisfying the same conditions of ~~sets~~ constraint as the set from which it is obtained. Furthermore we shall show that if a well-ordered set of continuous functions is of such a nature that any infinite subset contains at least one uniformly convergent sequence, the set will be constrained. ~~As a~~ It will also be shown that there is no denumerable set of subsets into which all monotone functions can be divided in such a manner that any ~~sub~~ subset of one of the subsets of the denumerable set contains a uniformly convergent sequence.

A set of continuous functions $\mathfrak{F}$ will be said to be constrained in an interval $a \leq x \leq b$ if

(1) $|f(x)| \leq A$ ~~it is bounded~~ in that interval, whenever f belongs to $\mathfrak{F}$;

(2) there is a function φ such that $\lim_{x \doteq 0} \varphi(x) = 0$, and whenever f belongs to $\mathfrak{F}$, and whenever $a \leq x_1 \leq b$, $a \leq x_2 \leq b$,

$$|f(x_1) - f(x_2)| \leq \varphi(|x_1 - x_2|).$$

We shall symbolize this state of affairs by

$$\mathfrak{F} \ \underset{a-b}{\text{const}} (A, \varphi).$$

Our first theorem then reads:

<u>Theorem</u> I. <u>If</u>

$$\mathfrak{F} \ \underset{a-b}{\text{const}} (A, \varphi),$$

<u>and</u> $\mathfrak{G}$ <u>is an infinite subset of</u> $\mathfrak{F}$, <u>then</u> $\mathfrak{G}$ <u>contains at least one sequence of distinct functions that converges uniformly to a limit</u> f <u>such that</u>

$$f \ \underset{a-b}{\text{const}} (A, \varphi).$$

<u>Proof</u>. Clearly

$$\mathfrak{G} \ \underset{a-b}{\text{const}} (A, \varphi).$$

Let $p_1, p_2, \ldots, p_n, \ldots$ be the set of all rational numbers between ~~A an~~ a and b, arranged in a progression. Then by Fréchet's theorem*, there is a

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sequence of functions $f_1, f_2, \ldots, f_n, \ldots$ belonging to $\mathfrak{C}$ ~~such~~ that for every p_k, the sequence $f_n(p_k)$ approaches a unique ~~limit~~ limit, which we may call $f(p_k)$. Suppose that the maximum ~~distance~~ difference of two adjacent p_l's for $l \le k$ is $D(k)$; clearly $\lim_{k=\infty} D(k)=0$. ~~Then max~~ Let $\max_{m \ge n,\, l \le k} |f_m(p_l) - f(p_l)| = \epsilon_{n,k}$. Then

$$\max_{m \ge n,\, l} |f_m(p_l) - f(p_l)| \le \epsilon_{n,k} + 2\varphi(D(k)).$$

~~Let us write $\epsilon_n = \epsilon_{n,n} + 2\varphi(D(n))$.~~

~~Since $\lim_{k=\infty} D(k) = 0$~~ Clearly if $n \ge n'$, $\epsilon_{n,k} \le \epsilon_{n',k}$, and $\lim_{n=\infty} \epsilon_{n,k} = 0$. ~~Let us call~~ Hence we may pick out a sequence $\epsilon_{n_1,1};\ \epsilon_{n_2,2}, \ldots, \epsilon_{n_k,k}$, such that $\lim_{k=\infty} \epsilon_{n_k,k} = 0$, while $n_1 \le n_2 \le \cdots \le n_k \le \cdots$

Call ~~$\epsilon_{n_k,k}$ ϵ_k~~. $\epsilon_{n_k,k} + 2\varphi(D(k))$ by the name ϵ_k; then $\lim_{k=\infty} \epsilon_k = 0$. Furthermore, for all $m \ge n_k$,

$$|f_m(p_l) - f(p_l)| \le \epsilon_k.$$

Hence for all rational values of x between a and b, $f_n(x)$ converges uniformly to $f(x)$.

If x is irrational, it is the limit of a sequence of rational numbers $x_1, x_2, \ldots, x_n, \ldots$

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Definition. A set of functions $\mathfrak{F}$ will be said to be constrained in an interval $a \leq x \leq b$ if

(1) it is bounded in that interval;

(2) there is a function φ such that $\lim\limits_{x \doteq 0} \varphi(x) = 0$, and ~~if~~ wherever f belongs to $\mathfrak{F}$, and whenever $a \leq x_1 \leq b$, $a \leq x_2 \leq b$,

$$|f(x_1) - f(x_2)| \leq \varphi(|x_1 - x_2|).$$

Theorem I. Any uniformly convergent sequence of continuous functions, together with its limit, forms a constrained set.

Theorem II. Any well-ordered set $\mathfrak{F}$ of continuous functions such that any infinite subset contains a sequence of distinct functions uniformly convergent in the interval $a \leq x \leq b$ is constrained in that interval.

Theorem III. If $\mathfrak{F}$ is a well-ordered constrained set of continuous functions in the interval $a \leq x \leq b$, and $\mathfrak{G}$ is ~~an~~ infinite sub-set of $\mathfrak{F}$, then $\mathfrak{G}$ contains at least one sequence of distinct functions that is uniformly convergent in the interval $a \leq x \leq b$.

Theorem IV The set of all monotone continuous functions cannot be divided into a denumerable number of constrained sets.

Proof of Theorem I. Let $\{f_n(x)\}$ be a uniformly convergent sequence of funstions in the interval $a \leq x \leq b$, and let $f(x)$ be its limit. Clearly if $n \geq k$, we have

$$|f_n(x) - f(x)| \leq \varepsilon_k,$$

where ε_k is a decreasing function of k that approaches 0 with k and is independent of x. There is consequently some k such that the set consisting of $f_n(x)$ $(n \geq k)$ and $f(x)$ ~~is less tha~~ contains no function whose maximum absolute value is greater than $\varepsilon_k + \max|f(x)|$. Our set of functions thus consists of a bounded set together with a finite set, and hence is bounded.

It remains to prove the existence of a φ. Let us make the definitions

$$\psi(y) = \max|f(x+z) - f(x)|, \text{ and}$$
$$\psi_n(y) = \max|f_n(x+z) - f_n(x)|,$$

where the maximum is taken for all values of x and $x+z$ lying between a and b, and for all values of z not greater than y. Since f and f_n are continuous, and hence uniformly continuous,

$$\lim_{y \doteq 0} \psi(y) = \lim_{y \doteq 0} \psi_n(y) = 0$$

Furthermore, ψ and ψ_n are by definition increasing functions.

We are now in a position to define φ. Let z_k be the largest number such that $\psi_n(z_k) \leq \varepsilon_k$ for all n up to and including $k-1$. Since there is only a finite number of such ψ_n's, and since $\lim_{y \doteq 0} \psi_n(y) = 0$, it follows that such a z_k exists and is not 0. It follows from the definition of ε_k that for $n \geq k$,

$$\begin{aligned} \psi_n(z_k) &= \max|f_n(x+z_k) - f_n(x)| \\ &= \max|f_n(x+z_k) - f(x+z_k) + f(x+z_k) - f(x) + f(x) - f_n(x)| \\ &\leq \max|f_n(x+z_k) - f(x+z_k)| + \max|f(x+z_k) - f(x)| + \max|f(x) - f_n(x)| \\ &\leq 2\varepsilon_k + \psi(z_k) \end{aligned}$$

We shall define $\varphi(y)$ as $2\varepsilon_k + \psi(z_k)$, where z_k is the first of the z's larger than y. We have just proved that $\psi_n(y)$ is less than or equal to $\varphi(y)$ for all y; it is obvious that $\psi(y) \leq \varphi(y)$. Hence

$$|f(x_1) - f(x_2)| \leq \varphi(|x_1 - x_2|)$$
$$|f_n(x_1) - f_n(x_2)| \leq \varphi(|x_1 - x_2|)$$

for every x_1 and x_2 in our interval. Furthermore,

$$\lim_{y \doteq 0} \varphi(y) = \lim_{k \doteq \infty} [2\varepsilon_k + \psi(z_k)] = \lim_{k \doteq \infty} 2\varepsilon_k + \lim_{z_k \doteq 0} \psi(z_k) = 0.$$

This completes the proof of our theorem.

<u>Proof of Theorem II</u>. It may be demonstrated at once that $\mathfrak{F}$ is a bounded set: if it is not, it can be divided into a denumerable set of subsets $\mathfrak{F}_n$, such that if $f_n(x)$ belongs to $\mathfrak{F}_n$, $n \le \max_x |f_n(x)| \le n+1$. Since $\mathfrak{F}$ is well-ordered, we can pick out a sequence $f_1(x), f_2(x), \dots, f_n(x), \dots$, from $\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n, \dots$ respectively. Then by theorem I, this sequence will not be uniformly convergent.

~~Since $\mathfrak{F}$ is bounded, for every x, there is an upper bound to $f(x)$ for the f's in $\mathfrak{F}$.~~

Let us write $\bar{f}(x)$ for $\max|f(x+y)-f(y)|$ for all positive x's and all ~~positive~~ x's and $x+y$'s lying between a and b. Let us write $\bar{\mathfrak{F}}$ for the class containing $\bar{f}$ when and only when $\mathfrak{F}$ contains f. $\bar{\mathfrak{F}}$ is clearly bounded. Hence for every x less than $|b-a|$ there is an upper bound to the values of $\bar{f}(x)$ for all $\bar{f}$'s belonging to $\bar{\mathfrak{F}}$. Call this $F(x)$. ~~Consider the following sets of functions~~ Either $\lim_{x \doteq 0} F(x) = 0$, and our theorem holds, or there is a sequence of values, $x_1, x_2, \dots, x_n, \dots$ such that $\lim_{n \doteq \infty} F(x_n) = a \neq 0$. Consider the following set of classes of functions: G_n consists of all functions g such that $|\bar{g}(x_n) - F(x_n)| \le \frac{1}{n}$. Clearly no G_n will be a null class. Now pick a g_n from each G_n. This can be done, since $\mathfrak{F}$ is well-ordered. ~~Clearly~~ Now, if

$$|g_n(x+x_n) - g_n(x)| \le \varphi(x_n), \text{ for } \cancel{\text{all } n} \text{ an infinity of } n\text{'s}$$

$\cancel{|\varphi(x_n) - F(x_n)| \le}$

$$\varphi(x_n) \ge F(x_n) - 1/n, \text{ for an infinity of } n\text{'s}$$

so that $\lim_{n \doteq \infty} \varphi(x_n) \neq 0$, or

$$\lim_{x \doteq 0} \varphi(x) \neq 0.$$

Hence by theorem I, the g_n's do not ~~form~~ contain a uniformly convergent sequence of functions. As this contradicts our hypothesis, it follows that $\lim_{x \doteq 0} F(x) = 0$, and F forms a function φ such that $\tilde{\Sigma}$ is constrained.

Proof of Theorem IV Let the rational points of the interval (a, b) be numbered $p_1, p_2, \ldots, p_n, \ldots$ Then the set of numbers of the form $f(p_1)$, where f ranges over all values in $\tilde{\Sigma}$, has at least one limit approached by an ∞ of distinct values of f. Let $f_{11}(p_1), f_{12}(p_1), \ldots, f_{1n}(p_1), \ldots$ be a set of values approaching the limit l_1. Consider the set of values

$f_{11}(p_2), f_{12}(p_2) \ldots, f_{1n}(p_2) \ldots$ These approach at least one limit. Let l_2 be the least limit of this sort. Let $f_{21}(p_2)$ be the first