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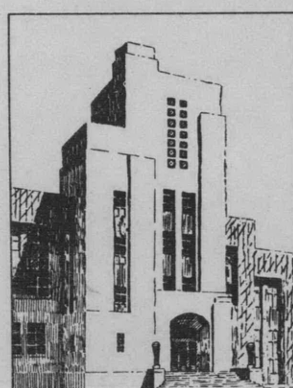
EXPERIMENTAL DETERMINATION OF THE HYDRODYNAMIC INCREASE IN MASS IN OSCILLATING BODIES

BY ROLF BRÄHMIG



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EXPERIMENTAL DETERMINATION OF THE HYDRODYNAMIC INCREASE IN MASS IN OSCILLATING BODIES

(DIE EXPERIMENTELLE BESTIMMUNG DES HYDRODYNAMISCHEN MASSENZUWACHSES BEI SCHWINGKÖRPERN)

by

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EXPERIMENTAL DETERMINATION OF THE HYDRODYNAMIC INCREASE IN MASS IN OSCILLATING BODIES

ABSTRACT

The methods for determining virtual mass under conditions of linear and vibratory motion known up to the present are briefly described. For the study and numerical representation of purely hydrodynamic inertia effects on oscillating bodies, a simple test method for forced vibrations is reported and illustrated by tests on vibrating disks. In converting the results of model tests to full scale, principles of similitude must be considered. These are given for the simplified case of a ship's propeller vibrating harmonically parallel to the free surface in an undisturbed fluid.

Due to streamline flow, a body moved in an inert medium undergoes an apparent increase in its "mechanical" mass by the amount of the "hydrodynamic" component of the total mass.* For an exact understanding of the forces or loads acting on the body, a knowledge of hydrodynamic inertia effects, and, in individual cases, their numerical magnitudes, is indispensable. In oscillatory phenomena in particular, the effect of the size of the oscillating mass on both the frequency and amplitude is worthy of note.

Numerous studies exist for the determination of the virtual mass as the sum of its components, the mechanical mass, and the hydrodynamic mass. This problem, which is highly important in many branches of engineering, is the main object of many of these studies. In numerous other papers it is treated as a subsidiary part of the main theme. On the basis of the method used to determine the hydrodynamic increase in mass, this specialized research can be subdivided as indicated in Table 1. Since the object of the present study is a more detailed treatment of test method IIb, the methods given under I. will be given only cursory attention.

In hydrodynamics the virtual mass, or the hydrodynamic increase in mass, can be determined by purely analytical methods if certain assumptions hold and if the bodies are of certain shapes. A body moving at constant velocity in an ideal medium, assumed to be infinite and at rest in infinity, encounters no resistance. The streamline flow about the body then corresponds to a steady potential flow with a distribution of pressure which produces no drag (d'Alembert's Paradox). In contrast, a body moving at a variable velocity, even in a condition of potential flow, meets with resistance which exceeds the pure mechanical mass resistance by the amount of the

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* Bessel, in the year 1828, was probably the first to call attention to this effect of the medium surrounding the moving body.

TABLE 1

Type of Motion	Method Used to Determine the Hydrodynamic Increase in Mass
I. No change in the direction of the moving body	1. Analytical 2. Experimental a) using the basic law of motion: force = mass x acceleration b) by oscillating tests
II. Periodic change in the direction of the moving body (oscillations)	1. Experimental a) from the natural frequency or the resonance frequency of a freely oscillating system b) at an arbitrary frequency by forced oscillations

inertia of the entrained mass of the medium. This hydrodynamic component of the mass can be calculated whenever the kinetic energy of flow or the change in momentum of flow, which can be determined for some simple geometric bodies by the theory of potential flow, is given. Various methods of calculation and their results are given in Lamb's (1)* basic work, supplemented by the studies of Lewis (2), Taylor (3), Tollmien (4), and others. For example, by using Rankine's source-sink method, which is valid only for axially symmetrical bodies, the coefficient of hydrodynamic inertia can be calculated from the product of the static moment of the respective sources and sinks and the density of the medium, as set forth by Munk (5).

The analytically determined values of the hydrodynamic increase in mass which, strictly speaking, are valid only for bodies in translational motion through an ideal medium, are used almost without exception as a basis for the calculation of both the elastic and inelastic vibrations of ship forms (2) (6) (7) (8) (9). In aerodynamics the concept of the "reduced wing mass" has been introduced into the study of the vibrations of airplane wings. In this there is likewise contained the value of the hydrodynamic increase in mass per unit length for an infinitely long plate of finite breadth in translational motion as calculated from potential theory (10) (11).

Whereas the calculated hydrodynamic mass depends only on shape, its values may vary with flow conditions in a real, eddying medium. A satisfactory agreement of the calculated result with the mass increase in the actual

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* Numbers in parentheses indicate references on page 19 of this translation.

flow is therefore possible only when the flow patterns of the two differing phenomena are identical.

The virtual mass of completely submerged or floating bodies in translational motion is determined experimentally by measuring the force of acceleration and the acceleration itself. As examples, the experiments by Abell (12), Baumann (13), and von den Steinen (14) are mentioned. It is a disadvantage that the data thus obtained combine resistance due to inertia and that due to friction. Since frictional resistance varies with time in accelerated motion, it is impossible to separate the two components. Hence, it is not possible to prove that the pure hydrodynamic inertial resistance is a function of the acceleration as it is suspected to be.

The method of determining the hydrodynamic increase in mass of a full-scale original in translational motion by comparative vibration tests with a geometrically similar model is based on the fact that the streamline flow about the oscillating body approaches potential flow, provided that motion is assumed to be translational. According to Föttinger (15) (16), this is true of oscillations of small amplitude and high frequency. If the virtual mass of the test form can be analytically determined, test results can be compared with calculated values for the inertia. Thus, for instance, the effect of finite size, rounded edges, and the like, which in general cannot be calculated for simple body shapes, can be checked. Tests on the vibration of rectangular plates which were made by Pabst (17) are an example of this.

The hydrodynamic mass increase of a vibrating body can be determined in terms of the free vibrations of the model. By this method the virtual mass of the model is determined from the difference between the natural frequency of the system in the test medium and the frequency for a motion completely devoid of losses.

$$m_h = \frac{c}{4\pi^2}\left(T_D^{\,2} - T_O^{2}\right)$$

where m_h is the hydrodynamic increase in mass in kg × sec^2 × cm^{-1},

c is the spring constant in kg/cm,

T_D is the period of the natural vibration of the freely vibrating damped test system, in seconds, and

T_O is the period of the oscillating undamped system, in seconds.

The equation which is also valid for flexural and torsional vibrations can be used for oscillations in liquid and gaseous media. Two arrangements are possible in liquids, i.e., the body may be submerged or floating. If the force of buoyancy acts in the direction of oscillation, it must be also taken into account in the case of floating bodies; see Reference (18).

The literature subsequently quoted represents only a modest selection from the large number of experiments which deal with the hydrodynamic mass increase of bodies in natural modes of vibration. For example, Bessel (19) determined the effect on the period of the internal friction and the inertia of the entrained medium by pendulum tests. Stokes (20) derived a theory for the oscillations of a sphere in a viscous medium whose validity he checked by Bessel's pendulum test. The oscillations of a sphere in a fluid, as well as the damping and the entrained mass of the medium, are the subject of numerous studies. To mention only the more recent studies there are those by Subrahmanyam (21), who summarizes the tests on spheres reported up to about 1937, and similar tests in Germany by Erbach, which Weinblum (7) quotes extensively. In the field of aerodynamics, studies by Pabst (17), Boccius (22), and Pleines (23), which contain further bibliographical material and references, are worthy of mention.

With reference to the investigations on the hydrodynamic coefficient of freely-oscillating bodies, which are evaluated according to the foregoing equation or its derivatives, it must be remarked that these formulas are based on the assumption of constant damping. In a strict sense they are valid for only this case. Often, however, the damping force does not increase linearly with the frequency, or it may be that the law of damping is not known. Even if damping has small effect on the observed quantity of the formula, the period T, the damping still exerts a considerable effect on the amplitude, for the decay of the free oscillations is governed by the attendant damping alone. Since, however, the flow pattern about an oscillating body and concurrently the magnitude of the entrained mass of the medium change not only with respect to frequency but also with respect to amplitude, the determination of the apparent mass by the method of free vibrations is inherently unreliable.

To what extent pure hydrodynamic forces of inertia affect an oscillatory phenomenon cannot be ascertained successfully in free damped oscillations, since the measured data contain both the damping resistance and the total (mechanical plus hydrodynamical) mass resistance. For example, the tests by Erbach (7) which have already been mentioned, and also those by Dimpker (18) and Holstein (24), must be evaluated in terms of this observation. Dimpker and Holstein studied the free-damped vibrations of a wedge, a cylinder, and a cube on the surface of water with respect to damping and entrained mass of water. They found that the apparent mass is a function of frequency, while Erbach's tests on the vibration of a sphere and an ellipsoid established that the hydrodynamic mass varies with maximum amplitude, although the variation is small. An answer to this problem would be welcome. The method frequently used of determining the natural frequency of the vibratory system from the impulse number of the system at resonance under periodic excitation instead of by free vibration tests (variable amplitude) seems suited to this purpose. The frequency (resonance frequency) which produces the maximum amplitude is then the natural frequency of the system. The hydrodynamic increase in mass is then determined by the equation previously stated. In this way, for example, Moullin and Browne (25) investigated the shifts of the natural frequencies of prismatic bars caused by entrained water, when the bars were completely immersed and the vibrations (2-node vibrations) took place parallel to the surface.* Hayes and Klein (27) performed vibration tests on the blades of full-scale marine propellers in the same way. In a manner similar to Moullin's tests, Schadlofsky (28) used periodic excitation to determine the natural frequencies of immersed steel vanes vibrating normal to the surface, in order to find the magnitude of the entrained mass of water. The advantage of this method compared to the free vibration test is that both the amplitude and the flow about the vibrating body remain constant. In this way the natural frequency of an immersed body as a function of the amplitude could still be studied, although it must be remembered that the effect of pure damping on the test result is not thus eliminated.

Free vibration tests have the disadvantage that they restrict the frequencies to a narrow range. The frequency in free vibrations is always the natural frequency of the vibrating system which is determined mainly by the vibrating mass and the spring constant. Hence, low test frequencies can only be achieved by relatively large oscillating masses in which case the effects of hydrodynamic inertia may become barely noticeable. Similarly, the production of high frequencies has limitations, since that restricts the initial amplitudes to very small values.

The result of this compulsory limitation of the test range in free vibrations is that the coefficients of hydrodynamic inertia determined by vibration tests on models at one or more frequencies were transferred to the prototype without considering the actual frequencies of its vibrations, i.e., without attention to the conditions of dynamic similitude of the two phenomena compared. It is to be emphasized that the model tests must correspond as closely as possible to the full-scale working conditions. Only then can reliable results be expected.

It has been shown in the foregoing that it is impossible to study the forces of hydrodynamic inertia acting on vibrating bodies separately from

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* Similar tests have been made by Browne, Moullin, and Perkins on prismatic bars immersed in water and considered as rigid bodies vibrating vertically with respect to the surface. See Reference (26).

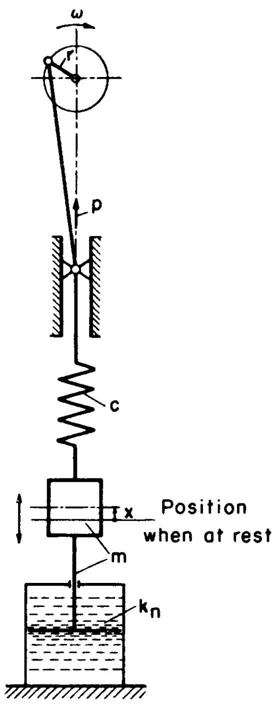

Figure 1 - System Designed to Produce Damped, Forced Vibrations

the effect of damping. For this purpose the attempt must be made to produce stationary streamlines about the oscillating body by forced vibrations.* Moreover, the test setup should be planned to permit determination of the entrained mass of the medium at any given amplitude and frequency of any given system, neglecting the damping force.

In the following a simple comprehensive method is given which satisfies these requirements to a large extent.

FUNDAMENTALS OF THE METHOD OF FORCED VIBRATIONS

For a system designed to produce damped forced vibrations, see Figure 1, the differential equation of vibrations can be solved exactly only for the special case of a linear differential equation with constant coefficients. Beyond this, approximations must be employed. These, however, are sufficiently accurate for many practical cases. In such calculations the damping force of the vibratory system is assumed to be replaced by an equivalent linear resistive force which during the period T consumes the same amount of energy as the true damping force (29) (30) (31). In these considerations the damping forces are usually assumed in the form $k_n\ (dx/dt)^n$. The differential equation of such a damped forced vibration is

$$m\frac{d^2x}{dt^2} + k_n\left(\frac{dx}{dt}\right)^n + cx = P\sin\omega t \qquad [1]$$

where m is the vibrating mechanical mass,**

x is the instantaneous displacement,

a is the peak value of the amplitude,

ϕ is the angle of phase,

t is the time,

k is the damping factor, or resistance for unit velocity,

c is the spring constant,

r is the crank radius of the exciting mechanism

* Owing to the constant maximum amplitude in this type of vibration, the maximum acceleration, which is the determining factor for continuous "inflow," is also constant and the flow pattern remains unchanged with respect to time.

** Including the entrained component of the spring mass.

P is the peak value of the exciting force,
ω is the frequency of vibration, or the frequency of excitation, and
ω_0 is the natural frequency of the undamped vibratory system.

The linear resistance term $k_1\ (dx/dt)^1$ is equivalent to the true damping force $k_n\ (dx/dt)^n$ if it leads to the same dissipation loss per cycle;

$$4k_n \int_0^{\frac{T}{4}=\frac{2\pi}{\omega}} \left(\frac{dx}{dt}\right)^n \left(\frac{dx}{dt}\right) dt = 4k_1 \int_0^{\frac{T}{4}=\frac{2\pi}{\omega}} \left(\frac{dx}{dt}\right)^1 \left(\frac{dx}{dt}\right) dt \qquad [2]$$

The assumption that a small damping force has negligible effect on the sinusoidal form of the vibration permits the statement

$$x = a\sin(\omega t - \phi) \qquad [2a]$$

and thereby an integration of Equation [2]. From this the damping factor of the equivalent linear damped vibration is

$$k_1 = k_n \gamma_n a^{n-1} \omega^{n-1} \qquad [3]$$

Substitution into Equation [1] gives

$$m\frac{d^2x}{dt^2} + k_n \gamma_n a^{n-1} \omega^{n-1} \frac{dx}{dt} + cx = P\sin\omega t \qquad [4]$$

where

$$\gamma_n = \frac{1}{2^n} \frac{\Gamma(n+2)}{\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}$$

is a numerical function depending only on the exponent n. Its values may be obtained from Figure 2.

The amplitude a and frequency ω are constant for the steady forced vibration. Therefore Equation [4] becomes a linear differential equation of familiar type whose solution is

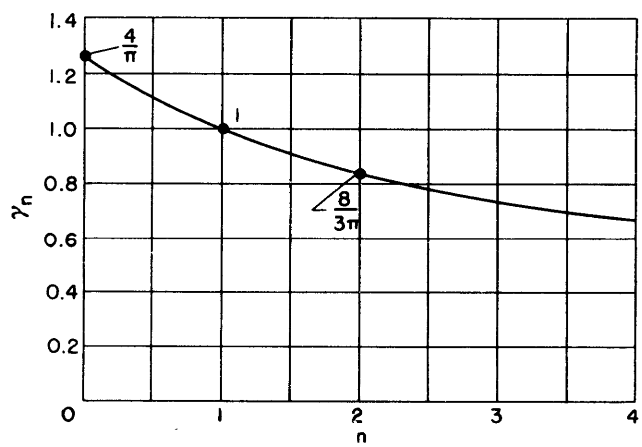

Figure 2 - The factor γ_n as a Function of the Damping Exponent n

$$x = \frac{P\sin(\omega t - \phi)}{\sqrt{(c - m\omega^2)^2 + k_n^2 \gamma_n^2 a^{2n-2} \omega^{2n}}} \qquad [5]$$

After equating Equations [2a] and [5]

$$a = \frac{P}{\sqrt{(c - m\omega^2)^2 + k_n^2 \gamma_n^2 a^{2n-2} \omega^{2n}}} \qquad [5a]$$

or

$$(c - m\omega^2)^2 a^2 + k_n^2 \gamma_n^2 a^{2n} \omega^{2n} = P^2 \qquad [6]$$

When $n = 1$, Equation [6] reduces to the exact solution of the forced vibration with a damping force proportional to the first power of the frequency,

$$(c - m\omega^2)^2 a^2 + k_1^2 a^2 \omega^2 = P^2 \quad \text{(Figure 3)} \qquad [6a]$$

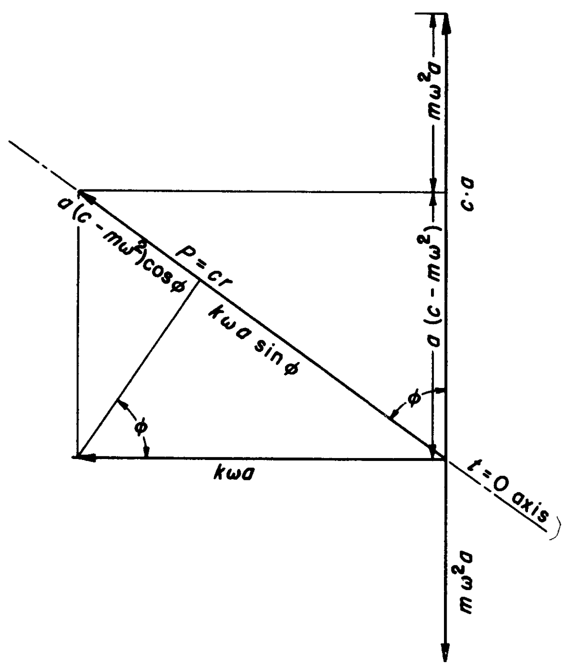

Figure 3 - Vectorial Analysis of Equation [6a]

$m\omega^2 a$ = Peak value of inertia
ca = Peak value of elastic force
$k\omega a$ = Peak value of damping force
P = Peak value of exciting force

For values of n other than 1, the agreement of the approximate and rigorous solutions is best when n is close to 1. When $n = 0$, i.e., for a damping force which does not depend on frequency (Coulomb's friction), the equivalent damping factor which must be substituted into the differential equation for vibration, Equation [1], is

$$k_0 \gamma_n a^{-1} \omega^{-1} = k_0 \frac{4}{\pi} a^{-1} \omega^{-1} \qquad [7]$$

When $n = 2$, (quadratic or hydraulic damping)

$$k_2 \gamma_2 a^1 \omega^1 = k_2 \frac{8}{3\pi} a \omega \qquad [7a]$$

If several damping forces act on the vibratory system, each of which is an arbitrary rational function of the frequency

$$k_{n_1}\left(\frac{dx}{dt}\right)^{n_1}, \; k_{n_2}\left(\frac{dx}{dt}\right)^{n_2} \ldots$$

then a superposition of the dissipation losses leads to an equivalent damping factor and the corresponding differential equation can again be solved.

One can derive from the approximate solution, Equation [6], which holds for an arbitrary law of damping, and which contains the amplitude a, an equation for the hydrodynamic increase in mass if certain conditions are observed in the vibration tests.

If two such tests using different values of c, m, ω, a, and P are made on a given vibratory system, Figure 1, the following equation is valid for Test 1

$$(c_1 - m_1'\omega_1^2)^2 a_1^2 + k_{n_1}^2 \gamma_{n_1}^2 a_1^{2n_1} \omega_1^{2n_1} = P_1^2 \quad [8a]$$

and for Test 2 the equation

$$(c_2 - m_2'\omega_2^2)^2 a_2^2 + k_{n_2}^2 \gamma_{n_2}^2 a_2^{2n_2} \omega_2^{2n_2} = P_2^2 \quad [8b]$$

holds, where m_1' and m_2' respectively denote the total (virtual) masses participating in the vibration, i.e.,

$$m_1' = m_1 + m_{h_1} \quad [9a]$$

$$m_2' = m_2 + m_{h_2} \quad [9b]$$

where m_1, m_2 are the mechanical masses of the vibratory system, and

m_{h_1}, m_{h_2} are the hydrodynamic increases in mass due to entrained fluid.

The damping force differs for each of the two tests, since it is a function of amplitude and frequency. However, identical damping conditions can be achieved, if the amplitude of the mass m and the frequency are the same in both tests. This condition can be met by an adjustment of the remaining parameters which do not affect the damping. Therefore

$$a_1 = a_2 = a = \text{constant}$$

$$\omega_1 = \omega_2 = \omega = \text{constant}$$

and

$$n_1 = n_2 = n$$

Thus

$$k_{n_1}^2 \gamma_{n_1}^2 a_1^{2n_1} \omega_1^{2n_1} = k_{n_2}^2 \gamma_{n_2}^2 a_2^{2n_2} \omega_2^{2n_2}$$

Then, by equating Equations [8a] and [8b], it follows that

$$(c_1 - m_1'\omega^2)^2 - \frac{P_1^2}{a^2} = (c_2 - m_2'\omega^2)^2 - \frac{P_2^2}{a^2} \quad [10]$$

Observing $P = cr$ and Equations [9a] and [9b], the expression

$$m_h = \frac{\dfrac{c_2^2 r_2^2 - c_1^2 r_1^2}{a^2} + (c_1^2 - c_2^2) + 2\omega^2(c_2 m_2 - c_1 m_1) + \omega^4(m_1^2 - m_2^2)}{2\omega^2[(c_1 - c_2) + \omega^2(m_2 - m_1)]} \quad [11]$$

is obtained for the hydrodynamic increase in mass. Therefore, Equation [11] is valid in general for all laws of damping for which there is a solution of the vibration equation in the form of Equation [6] such that the factors which appear in the damping term can be retained unchanged in each pair of tests. Hence, no test variables may appear in this term. Thus Equation [11] which determines m_h also remains unchanged when the vibratory system is subjected to a "mixed damping," for instance to a constant damping force (Coulomb's friction) and to a damping* proportional to the first power of the frequency. This case is important from the standpoint of testing technique insofar as a supplementary control of the free end of the vibrating spring may become necessary under certain conditions to prevent disturbing secondary movements.

Let it be further mentioned that the method given herein for the calculation of m_h permits use of the ***implicit*** form of the solution, Equation [6], of the differential equation for vibration. Hence, the absolute magnitude of the amplitude a does not have to be known. This constitutes a considerable advantage, since the solution of the explicit form, which is simple for linear damping, i.e., when $n = 1$, becomes quite involved for quadratic damping, i.e., when $n = 2$. Equation [6] can be solved only by approximations for n exponents, that is, for exponents other than 1 and 2.

The vibratory condition required by parallel tests ($a_1 = a_2 = a =$ constant, when $\omega =$ constant) can also be more simply achieved experimentally by varying only one of the quantities r, m, or c. From Equation [11] it is immediately apparent that when

$$r_1 = r_2 = r = \text{constant} \ldots\ldots m_h = f(c_1, c_2, m_1, m_2),$$

$$m_1 = m_2 = m = \text{constant} \ldots\ldots m_h = f(c_1, c_2, r_1, r_2),$$

$$c_1 = c_2 = c = \text{constant} \ldots\ldots m_h = f(r_1, r_2, m_1, m_2).$$

In the method frequently used of exciting vibrations by a crank drive, the variation of the crank throw r, and the variation of the vibrating mass m, while the vibrating spring is kept constant, is probably the most convenient. For this case Equation [11] assumes the simpler form

$$m_h = \frac{c}{\omega^2}\left[\frac{c}{2a^2\omega^2}\;\frac{r_2^{\,2} - r_1^{\,2}}{m_2 - m_1} + 1\right] - \frac{m_1 + m_2}{2} \qquad [12]$$

VIBRATION TESTS ON DISKS

Tests by the method developed for the system shown in Figure 1 were performed on a longitudinal vibrating system with disks as vibrating bodies.

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* The equivalent damping factor for mixed damping is

$$k_1 = k_0 \frac{4}{\pi}\, a^{-1}\omega^{-1} + k = \frac{4}{\pi}\, R\, a^{-1}\omega^{-1} + k$$

where R is a constant force of friction. Equation [6] must now be modified to

$$(c - m\omega^2)^2 a^2 + \left(\frac{4}{\pi}\, R\, a^{-1}\omega^{-1} + k\right)\omega^2 a^2 = P^2$$

The hydrodynamic increase in mass m_h can here be calculated and a comparison of test results and calculated results is possible. The test setup is shown in Figure 4. A cylindrical spring F carries the oscillating mass at its lower extremity. This mass consists of the vibrating body S, its support and the variable supplementary masses Z. The disks are cut from plating of a thickness of 3 mm (0.12 inch) with sharp edges. Their diameter is 144, 176, and 205 mm (5.67, 6.93, and 8.07 inches). The vibrating disks are immersed in a vessel 520 mm (20.47 inches) in diameter filled with water. A constant mean depth of immersion h_m of 200 mm (7.87 inches) is maintained in all tests and the mean distance h_m' from the bottom of the vessel is 160 mm (6.30 inches). The upper end of the spring F is secured to a flat steel bar which is supported and guided by ball bearings on all faces. This steel bar in turn is joined to the crank-drive exciter E by a short connecting rod and the crank pin K. The crank throw or radius r of the exciter can be varied from 0 to 25 mm (0 to 0.98 inch) by shifting the adjustable crank pin K. The RPM of the drive and hence the exciting frequency can also be varied widely.

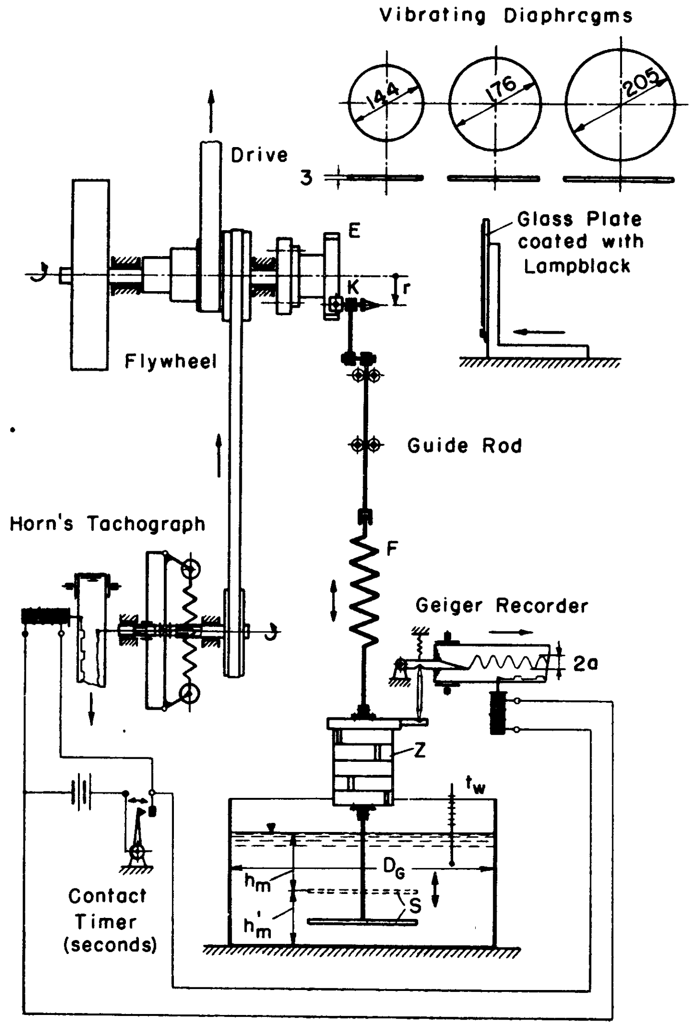

Figure 4 - Diagram of the Test Apparatus to Determine the Hydrodynamic Increase in Mass of Vibrating Disks

The vibrations were recorded by a Geiger Universal-Registriergerät (Geiger Universal-Recorder)* and the exact measurement of the amplitude a was effected with a microscope. The crank radius r of the exciter was microscopically determined from the records scratched on a glass plate covered with

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* Manufactured by Lehmann and Michels, Altona, Germany.

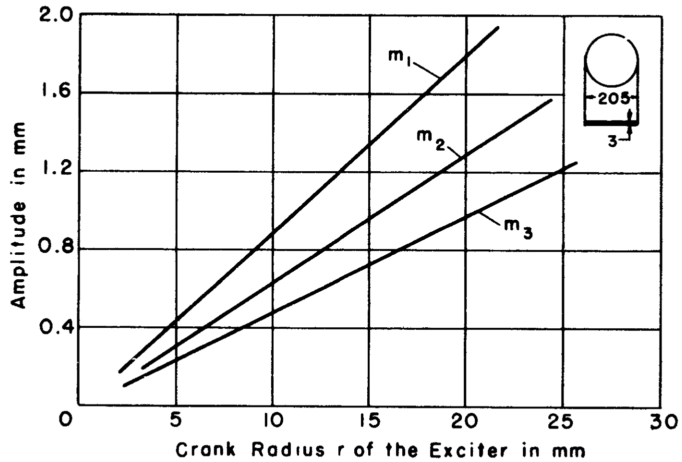

Figure 5 - Amplitude a as a Function of the Crank Radius r of the Exciter at Constant Frequency of Excitation for a Disk 205 mm (8.07 inches) Diameter

Frequency of excitation	ω	$= 65.24\ sec^{-1}$ = constant
Spring constant	c	= 1.792 kg/cm (14.45 lb/in)
Mean temperature of water	t_{w_m}	= 14.8 degrees Centigrade (27.4 degrees Fahrenheit)
Kinematic viscosity	ν	$= 1.1472 \cdot 10^{-2}\ cm^2/sec$
Mechanical vibrating masses	m_1	$= 2.1011 \cdot 10^{-3}\ kg \cdot sec^2/cm$
	m_2	$= 4.3662 \cdot 10^{-3}\ kg \cdot sec^2/cm$
	m_3	$= 6.6730 \cdot 10^{-3}\ kg \cdot sec^2/cm$
Diameter of vessel	D_g	= 520 mm (20.47 inches)
Mean depth of immersion	h_m	= 200 mm (7.87 inches)
Mean distance from bottom	h_m'	= 160 mm (6.30 inches)

lamp black by the revolving, pointed crank pin. The mean exciting frequency was derived from the RPM's of the driving gear which were recorded with a tachometer. Records made with a Horn Tachograph and the time-calibrated scriber recordings of the Geiger Recorder of the RPM controller, as well as a check of the random fluctuation of the RPM's, also assisted in determining the mean frequency of excitation. The spring constant c of the vibrating spring was determined not only by static loading, but by special vibration tests as well. The same was true of the mechanical vibrating masses affecting the calculation because of the vibrating spring mass component contained in them.

Figures 5 and 6 show the results of vibration tests on the disk whose diameter D was 205 mm (8.07 inches).

As previously stated, the vibration tests at constant frequency must be made in groups, with the amplitude kept constant. In test procedure, it is naturally more convenient to record the function of the variable crank radius r of the exciter at constant mechanical masses m and to read off the values of r for the various vibrating masses at a given value of a; see Figure 5.

The functional relationship $a = f(r)$, when m is constant, results directly from the tests. Using the notation $(r^2_{max} - r^2_{min}) = \Delta(r^2)$ and $m_{max} - m_{min} = \Delta m$, the equation for m_h, i.e., Equation [12] can be written as

$$m_h = \frac{c^2}{2\omega^4}\,\frac{\Delta(r^2)}{a^2\,\Delta m} + \left[\frac{c}{\omega^2} - \frac{m_{max} + m_{min}}{2}\right] \qquad [13]$$

This is an equation of a straight line: $y = px + q$ with $p = c^2/2\omega^4$ as slope. The graph of the equation permits a simple control of the m_h value, if the function $a = f(r)$ is determined for more than two mechanical vibrating masses

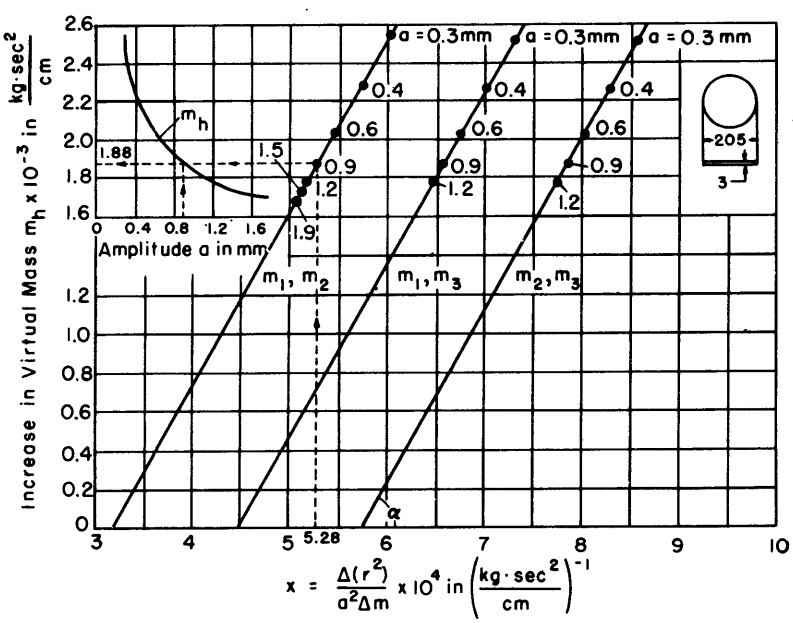

Figure 6 - Determination of the Virtual Mass m_h as a Function of the Amplitude a for a Disk 205 mm (8.07 inches) Diameter

Frequency of excitation	ω	$= 65.24$ sec^{-1} = constant
Spring constant	c	$= 1.792$ kg/cm (14.45 lb/in)
Mean temperature of water	t_{w_m}	$= 14.8$ degrees Centigrade (27.4 degrees Fahrenheit)
Kinematic viscosity	ν	$= 1.1472 \times 10^{-2}$ cm^2/sec
Mechanical vibrating masses	m_1	$= 2.1011 \times 10^{-3}$ kg · sec^2/cm
	m_2	$= 4.3662 \times 10^{-3}$ kg · sec^2/cm
	m_3	$= 6.6730 \times 10^{-3}$ kg · sec^2/cm

$$m_h = \frac{c^2}{2\omega^4}\,\frac{\Delta(r^2)}{a^2\,\Delta m} + \left[\frac{c}{\omega^2} - \frac{m_{\max} + m_{\min}}{2}\right]$$

$$\tan\alpha = \frac{c^2}{2\omega^4}$$

in conformity with Figure 5. It is then merely necessary to plot the value of the abscissa $x = \Delta(r^2)/a^2\Delta m$ to be able to furnish the desired values of m_h for any given amplitude a, see Figure 6.

The test results for the disks studied are given for an individual case in Figure 6 and summarized in Figure 7.

The strong functional relationship of the virtual mass to the amplitude and naturally to the diameter of the disk also, is noteworthy. The results uniformly give a survey of the behavior of a vibrating body immersed in a fluid. In an ideal fluid, the kinetic energy transferred to the vibrating body as a result of its streamline flow achieves its full-scale effectiveness, i.e., the virtual increase in mass reaches its maximum value. The true, viscous fluid is characterized by the formation of a boundary layer about the

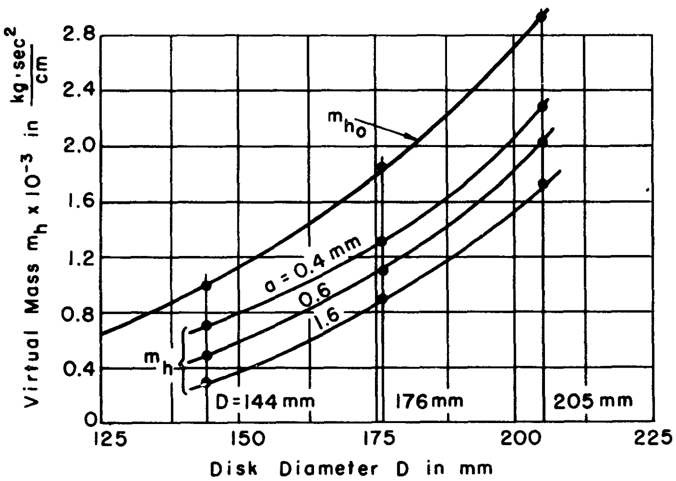

Figure 7 - Virtual Mass m_{h_0} or m_h for Disks as a Function of the Diameter in Water

$$m_{h_0} = \frac{8}{3}\rho r^3 = \frac{\rho}{3}D^3 \quad \text{Reference (1)}$$

where $\rho = 1.02 \times 10^{-6}$ kg · sec²/cm⁴ for water. This equation is valid for translational motion of an infinitely thin disk in an ideal fluid of infinite extent. m_h is obtained from test results with sharp-edged disks 3 mm (0.12 inch) thick, vibrating in water. The exciting system is shown in Figure 4. Test conditions conform to Figure 5.

body surrounded by flow, and its separation, accompanied by the formation of eddies. A part of the total kinetic energy is destroyed owing to the formation of eddies (damping). Hence, the hydrodynamic increase in mass must be smaller compared to that in the ideal fluid. At vibrations of small amplitude and high frequency, the dissipation of the boundary layer is extensively hindered. The flow about the vibrating body then approximately corresponds to the ideal case, i.e., it corresponds to potential flow about the body.* The kinetic energy, which the entrained fluid makes thus available, is then almost equal to its maximum value.

Exhaustive tests were only made at a frequency of $\omega = 65.24$ sec^{-1}. Individual tests at higher or lower values of ω confirmed the suspected effect of frequency on the magnitude of the hydrodynamic increase in mass.

In comparing test results with mathematical calculation, it must be considered that the latter are only valid for an infinitely thin disk in translational motion in an ideal fluid of infinite extent, whereas test results contain the effects of the proximity of the walls, the finite thickness of the disk, the possibility of elastic natural vibrations of the disks, the properties of viscous fluids, and other factors.

The foregoing method of determining the hydrodynamic increase in mass of vibrating bodies has proved easy and practicable and seems suited for use when testing ship's propellers which are familiarly known to be affected by hydrodynamic inertia forces.**

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* See the footnote on page 6.

** Studies by Guntzberger (32) and Baumann (33) represent the first attempts to find an explanation of hydrodynamic effects on ship propellers.

LAWS OF SIMILITUDE IN VIBRATION TESTS ON MODEL SHIP PROPELLERS

For the basically simplest case of a ship propeller in torsional vibrations, shown in Figure 8, where the plane of vibration is parallel to the surface of the fluid, Equation [11] is replaced by

$$\Theta_h = \frac{\dfrac{c_2^2\psi_2^2 - c_1^2\psi_1^2}{\phi^2} + (c_1^2 - c_2^2) + 2\omega^2(c_2\Theta_2 - c_1\Theta_1) + \omega^4(\Theta_1^2 - \Theta_2^2)}{2\omega^2\left[(c_1 - c_2) + \omega^2(\Theta_2 - \Theta_1)\right]} \quad [14]$$

where Θ_h is the hydrodynamic mass moment of inertia of the propeller with respect to its axis of vibration,

Θ_1, Θ_2 are the mechanical mass moments of inertia with respect to the axis of vibration of the propeller,

c_1, c_2 are the spring constants for torsional vibrations,

ψ_1, ψ_2 are the torque amplitudes of the excitation, in radians,

ϕ is the amplitude of the vibration of the propeller, in radians, and

ω is the frequency of the vibration or frequency of the exciter.

In the simpler form used for purposes of test procedure, the foregoing equation reads

$$\Theta_h = \frac{c^2}{2\omega^4}\,\frac{\Delta(\psi^2)}{\phi^2\Delta\Theta} + \left[\frac{c}{\omega^2} - \frac{\Theta_{max} + \Theta_{min}}{2}\right] \quad [15]$$

where $\psi_2^2 - \psi_1^2 = \Delta(\psi^2)$ and $\Theta_2 - \Theta_1 = \Delta\Theta$. Hence, this equation corresponds to Equation [13] which is valid for a longitudinal vibrating system. It is evident from previous discussion and also from Equation [15] that the determination of Θ_h requires at least two series of tests. Different mechanical mass moments of inertia must be used in each test. Moreover, still another simplification of test procedure can be achieved. Equation [15] contains the squares of the ψ-values. Therefore, the mechanical moments of inertia should be chosen for each test series so that the curves $\phi = f(\psi)$ which must be experimentally determined diverge as strongly as possible. If one of the two mechanical mass moments of inertia of the vibrating body is made so large that neither Θ_h nor the damping resistance affect the amplitude any longer, then $\phi = f(\psi)$ can be calculated for this test series from the familiar equation for the amplitude of torsional vibration of the undamped forced vibration; this is the equation of a straight line when

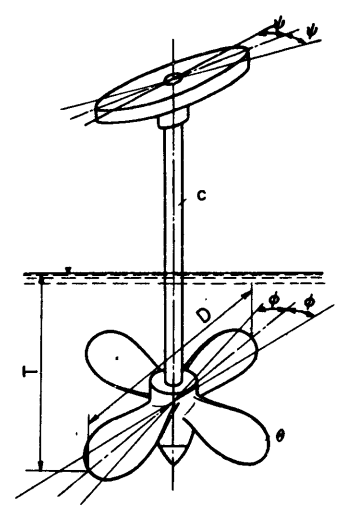

Figure 8 - Torsional Vibrating System with a Ship Propeller as Vibrating Body

Θ, ω, and c are constants.

$$\phi_2 = \frac{\psi_2}{1 - \frac{\Theta_2 \omega^2}{c}} = \text{constant} \times \psi_2 \qquad [16]$$

If the value of Θ_2, which makes the test curve $\phi_2 = f(\psi_2)$ coincident with the straight line of Equation [16], has been determined by preliminary tests, further vibration tests need to be made only with the smallest possible mass moment of inertia Θ_1, and the unknown Θ_h is then determined.

In transferring the numerical values of Θ_h obtained from tests on a model ship propeller to the prototype, certain conditions of similitude must be observed. For a propeller harmonically vibrating about its axis in still water, Figure 8, the following functional relationship of the variables exists in accordance with the laws of mechanical similitude established by Weber (34).

$$F(D, t, \rho, \Theta_h, \eta, \gamma, a, H. \ldots, T, \delta)^* = 0 \qquad [17]$$

where D is the diameter of the propeller in meters,

t is the period of a complete oscillation in seconds,

ρ is the density of the fluid in $m^{-4} \cdot sec^2 \cdot kg$,

γ is the specific gravity of the fluid in $m^{-3} \cdot kg$,

η is the physical viscosity of the fluid in $m^{-2} \cdot sec \cdot kg$,

Θ_h is the hydrodynamic mass moment of inertia in $m \cdot sec^2 \cdot kg$,

a is the maximum amplitude of vibration of the propeller blade tips in meters,

H is the pitch of the pressure surface of the propeller in meters,

T is the depth of immersion of the tips of the propeller blades in meters, and

δ is the absolute size of grain (texture, coefficient of roughness) on the propeller blade surface in meters.

Since this is a dynamic problem, three reference terms, for example, D, t, ρ (corresponding to the three basic units of dynamics) must be chosen, in terms of which the equation in dimensional quantities $F = 0$ is converted into an equation in non-dimensional exponential products, i.e., so-called characteristic coefficients Π. The material constants η and γ lead to special model laws, whereas a, H, T, and δ become non-dimensional parameters in the characteristic equation. The non-dimensional terms, which enter the equation $F = 0$, are

$$\Pi_1 = \frac{\Theta_h}{D^5 \rho} = \mathfrak{N} \quad \text{a form of Newton's number}$$

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* Additional quantities which determine the shape of the ship's propeller are omitted.

$\Pi_2 = \frac{\nu}{D^2 t^{-1}} = \Re^{-1}$ a form of Reynolds number*

$\Pi_3 = \frac{g}{D t^{-2}} = \Im^{-1}$ a form of Froude's number**

$\Pi_4 = \frac{a}{D} = \mathfrak{P}_a$ the amplitude parameter

$\Pi_5 = \frac{H}{D} = \mathfrak{P}_H$ the pitch parameter

$\Pi_6 = \frac{T}{D} = \mathfrak{P}_T$ the immersion parameter

$\Pi_7 = \frac{\delta}{D} = \mathfrak{P}_\delta$ the roughness parameter

The characteristic equation then becomes

$$\psi\left(\frac{\Theta_h}{D^5 \rho}, \frac{\nu}{D^2 n_s}, \frac{g}{D n_s^2}, \frac{a}{D}, \frac{H}{D}, \frac{T}{D}, \frac{\delta}{D}\right) = 0 \qquad [18]$$

where $n_s = \omega/2\pi = t^{-1}$ cycles per second. With $\mathfrak{N}$ or $\mathfrak{K}_{\theta_h}$ as the desired unknowns, Equation [18] finally becomes

$$\mathfrak{N} = \mathfrak{K}_{\theta_h} = \Phi(\Re^{-1}, \Im^{-1}, \mathfrak{P}_a, \mathfrak{P}_H, \mathfrak{P}_T, \mathfrak{P}_\delta) \qquad [19]$$

In order to attain complete dynamic similitude, the two numbers $\Re^{-1}$ and $\Im^{-1}$ depending on the properties of matter must be held constant for both model and prototype. This requirement can generally not be fulfilled, since the ratio of the kinematic viscosities of the mediums used is fixed as

$$\frac{\nu'}{\nu} = \sqrt{\lambda^3}$$

where† $\lambda = D'/D$ is the model scale, chosen at random. Hence, for a model scale of $\lambda = 14$, for example, a medium having a kinematic viscosity of about one-fiftieth that of the prototype (full-scale medium) should be used in model tests. Simultaneous observation of both Reynolds and Froude's Laws must be abandoned in favor of the latter in model tests, because of their mutually contradictory requirements. However, neglecting Reynolds Law will have but slight effect on the result, because the forces of inertia which satisfy Newton's Law of Similitude occurring in the phenomenon and the gravitational effects together far outweigh the frictional resistances.

For a model propeller of constant pitch H, geometrically similar to the prototype, the pitch ratio H/D remains unchanged, therefore the pitch

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* Kinematic viscosity $\nu = \eta/\rho$ ($m^2 \cdot sec^{-1}$).

** $g = 9.81 \ m \cdot sec^{-2}$.

† The prime factors represent the prototype.

parameter $\mathfrak{P}_H$ drops out of the characteristic equation. The immersion parameter $\mathfrak{P}_T$ can also drop out, if the coefficients $\Re_{\theta_h}$ obtained from model tests are related to the same immersion ratio T/D of all prototypes geometrically similar to the model propellers. In contrast, however, when studying the effect of the depth of immersion $\Re_{\theta_h}$, $\mathfrak{P}_T$ must be retained in the characteristic equation. The requirement of geometric similitude with respect to surface quality can be fulfilled approximately by fabricating model propellers with the finest, smoothest surface finish possible. Even if the fineness of the surface of the models cannot be scaled exactly to the value of the "relative roughness" δ/D of the prototype; the ideal roughness parameter $\mathfrak{P}_\delta$ can be considered nevertheless the same for both model and prototype as a first approximation, and hence may be omitted from the characteristic equation.

Complete dynamic similitude of the phenomenon of vibration cannot be attained. Therefore, the simplified expression for the characteristic function of hydrodynamic moment of mass inertia is

$$\Re_{\theta_h} = \phi(\mathfrak{F}^{-1}, \mathfrak{P}_a, \mathfrak{P}_T) \qquad [20]$$

In the comparative study of variously shaped models, Equation [20] should include additional supplementary parametric values characteristic of the propeller shape.

Model tests based on Equation [20] permit a non-dimensional expression of the hydrodynamic moment of mass inertia for a definite type of propeller as a function of the amplitude and frequency at various depths of immersion. If Θ_h is desired for a prototype similar to the scale model, the value $\Re_{\theta_h}$ must be read from the curves for the model tests at $\mathfrak{F}^{-1} = \frac{g}{D' n_s'^2}$, a'/D' and T'/D' of the prototype. From the latter it follows directly that

$$\Theta_h' = \Re_{\theta_h} D'^5 \rho' \qquad [21]$$

For completeness we shall add a few additional numerical values for the characteristic vibration $\mathfrak{F}^{-1}$ and for the relative peak amplitude a'/D' of propeller blades found for the propellers of full-scale marine engines.

Use of the $\mathfrak{F}^{-1}$ and a'/D' values given below require that the tests on the model propellers which have, for example, a diameter $D = 350$ mm (13.78 inches mean diameter) be performed at vibrational velocities of about n_s = 15 to 25 cycles per second and amplitudes of about $a = 9$ to 20 mm (0.35 to 0.79 inch) (arc length). The size of model propellers should be determined by two factors: first, by the vibrational frequencies and amplitudes attainable with the experimental setup; and second, by the fact that the largest possible propellers are advantageous for reasons of testing technique.

TABLE 2

		Dimension	Single-Screw Tow Boat	Single-Screw Freighter	Single-Screw Freighter
Ship speed	V	knots	10.5	11	13.5
Propeller RPM	n	RPM	250	160	115
Effective output of the engines	N_e	HP	350	950	3500
Propeller diameter	D'	m	1.85	2.95	4.70
Pitch of the propeller blades	H'	m	1.52	2.35	3.76
Number of propeller blades	z		3	4	4
Mechanical moment of propeller vibration	$(GD)'^2$	$kg \cdot m^2$	306	3190	23500
Natural frequency first order	n_I'	CPM	660	422	248
Propeller vibrations per second	n_s'	CPS	11	7.03	4.14
Characteristic frequency coefficient	$\mathfrak{F}^{-1}$		4.37×10^{-2}	6.73×10^{-2}	12.2×10^{-2}
Maximum amplitude of vibration of the propeller blade tips measured in resonance	a'	m	0.048	0.15	0.259
Relative peak amplitude of the propeller blades	a'/D'		0.026	0.0508	0.055

The extension of model tests to include periodic vibrations superimposed on the rotating propeller, as it occurs with the full-scale prototype, should offer no special difficulties. This will permit an investigation of the effect of the propeller flow or wake on these vibrations.

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