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SOME REMARKS ON STATIONARY SCHEMES FOR CAVITATION FLOW
ABOUT A FLAT PLATE

by

M.I. Gurevich

November 1948 Translation 224

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Some Remarks on Stationary Schemes for Cavitation Flow about a Flat Plate

(Nekotorye Zamechaniia o Statsionarnykh Skhemakh Kavitatsionnogo Obtekaniia Plastinki)

by

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Translation and Comments

by

J.V. Wehausen

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SOME REMARKS ON STATIONARY SCHEMES FOR CAVITATION FLOW ABOUT A FLAT PLATE*

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M. I. Gurevich

1. The appearance of cavitation flow.

It is known that behind bodies moving in water at sufficiently high velocities a cavity opens up. In Figure 1 is shown a sketch from a photograph of such a flow taken by Ackeret. Experiments indicate that the pressure inside the essential part of the cavity is constant. With constant speed of the body the flow in its neighborhood has a clearly defined stationary character which is disturbed only at the end of the cavity. The degree of development of cavitation is usually characterized by the "cavitation number"

$$(1.1) \quad \lambda = \frac{2(p_\infty - p_0)}{\rho\, v_\infty^2} = \frac{v_0^2}{v_\infty^2} - 1$$

where λ is the cavitation number, ρ is the density of water, p_∞ and v_∞ are the pressure and velocity at infinity, p_0 and v_0 the pressure in the cavity and the velocity on its boundary.

For small cavitation numbers, i.e., when a region free of water occurs, it is natural to try to devise a scheme for the cavitation flow about the body which will allow one to find the forces acting on the body and the dimensions outlining the cavity. Proceeding to the description of such schemes, we shall, for simplicity of exposition, speak only of schemes of cavitation flow about a flat plate of infinite aspect ratio, situated perpendicularly to the oncoming plane flow of a weightless ideal incompressible fluid. The generalization of the problem to the case of a wing profile with an arbitrary angle of attack brings up not so much difficulties in principle as mathematical difficulties.

2. The scheme of Betz.

Betz [1] proposed to apply Rayleigh's well-known theory of free boundary flow to the solution of the problem of cavitation (Figure 2). This, in particular, presents cavitation flow about the flat plate for a cavitation number $\lambda = 0$. In this

* Bulletin de l'Académie des Sciences de l' URSS, Classe des Sciences techniques, 1947, 143-150 (1947)

case the cavity has infinite dimensions and the velocity v_0 on its boundary equals the velocity at infinity. The drag coefficient of the plate is $C_x = 2X/\rho \ell v_\infty^2$, where X is the drag and ℓ is the length of the plate. For the case being considered by us, when the angle of attack is $\alpha = \pi/2$, the drag coefficient, as is known, is given by

$$(2.1) \qquad (C_x)_{\lambda=0} = \frac{2\pi}{\pi + 4} = 0.88$$

For a cavitation number not equal to zero Betz proposed to apply to the oncoming flow this same flow of Rayleigh and only take into consideration that the pressure on the back side of the plate will equal $p_0 < p_\infty$. In this manner, the drag coefficient of the plate becomes

$$(2.2) \qquad C_x = \frac{2\pi}{\pi + 4} + \frac{2(p_\infty - p_0)}{\rho V_\infty^2} = \frac{2\pi}{\pi + 4} + \lambda$$

A graph of the variation of C_x with λ (a straight line) is shown on Figure 11. Betz justifies his formula with the remark that even though the inequality of pressure between infinity and the cavity influences the form of the oncoming flow, the influence may be considered negligeable for small cavitation numbers. For large cavitation numbers the term $2\pi/(\pi+4)$ is itself small in comparison with λ and describes the influence of the flow pattern with less accuracy.

The scheme of Betz is useful for deriving an approximate expression for the resistance of a flat plate but it doesn't set up and solve a hydrodynamical problem. For a formally correct solution of the problem one must propose a scheme* with variable pressure or velocity along the streamlines issuing from the edges of the flat plate (Figure 3).

Consequently, we propose that along the above mentioned streamlines the velocity decreases from v_0 to $v_\infty < v_0$. The regions of variation of the complex potential w and of the functions $\zeta = \frac{v_\infty dZ}{dw}$ are given on Figures 4 and 5. The region of variation of w is the complex plane with a cut along the real axis from 0 to ∞. The cut represents the flat plate and the boundary streamlines. This region coincides exactly with the corresponding region for the Rayleigh flow. The region of variation of $v_\infty \, dZ/dw$ is the lower half-plane from which has been cut out a piece bounded by the line CEB corresponding to the boundary streamline.

The flat plate is transformed onto that part of the real axis for which $|\zeta| > v_\infty/v_0 = q$. The shape of the boundary

* The suggestion to examine this scheme was made by L. I. Sedov, Corresponding Member of the Academy of Sciences of the USSR.

BEC in the ζ-plane is unknown. Concerning this we know only that in the points C,E and B, $\zeta_B = -\zeta_C = q$ and $\zeta_E = -i$. Besides this, in the limiting case of the Rayleigh flow, when $q = 1$, the line CEB becomes a semi-circle.

We shall take the contour CEB as the arc of an ellipse with semi-axes q and 1. With such an assumption it is not difficult to solve the problem by means of conformal mapping of the regions of variation of ζ and w upon the lower half-plane of the variable t (Figure 6). The following formulas realize these transformations:

$$w = \frac{\ell v_\infty}{4qt\pi} \frac{1}{t^2} \tag{2.3}$$

$$v_\infty \frac{dZ}{dw} = qt + \sqrt{t^2-1} \tag{2.4}$$

where ℓ is the length of the flat plate, and q is expressed through the cavitation number by the formula

$$q = \frac{1}{\sqrt{1+\lambda}} \tag{2.5}$$

By means of the Bernoulli Theorem and the formulas (2.3), (2.4), and (2.5) one may find the drag coefficient of the flat plate which we shall, as before, designate by

$$c_x = 1+\lambda + \frac{4}{4\sqrt{1+\lambda} + \pi(1+\lambda)} \left[\frac{\pi}{4}(1-\lambda) - \sqrt{1+\lambda} + \sqrt{\lambda} \ln\left(\sqrt{1+\lambda} + \sqrt{\lambda}\right)\right] \tag{2.6}$$

Curve 1 of Figure 11 was obtained with the aid of (2.6). One may also consider another variant of the scheme just presented which consists in considering the arc of the ellipse not as a boundary in the ζ-plane but as the corresponding boundary in the plane of $\frac{1}{v_\infty}\frac{dw}{dZ}$ (Figure 7). These two schemes are not identical since the inversion of an ellipse is not an ellipse. The solution of the problem of mapping a semi-ellipse on to a half-plane is well known [2].

The drag coefficient is expressed by integrals which may be computed by numerical methods. For small cavitation numbers according to our calculations the curves of the drag coefficient plotted against the cavitation number for the two methods practically coincide.

3. The scheme of Riabouchinsky - Weinig

Let us proceed to the examination of a scheme which may be used not only to find the drag of a flat plate but also the dimensions of the region of the cavity. The solution of the problem of free boundary flow about two plates as shown in Figure 8 is due to D. Riabouchinsky [3], [4], [5]. The plates are identical and the flow pattern is symmetrical about the x-axis. The velocity at infinity is equal to v_∞. On the streamlines CMD and C'M'D' the speed is constant and equal to $v_0 > v_\infty$. When the distance between the plates is made equal to zero, one obtains the usual flow about a flat plate. When one of the plates recedes to infinity, the Rayleigh flow about the other plate obtains. Weinig [7] proposed to use as a model of the cavitation flow about one flat plate the scheme of Riabouchinsky, making the supposition that the inexact model of the flow near the end of the cavity, caused by the closing of the cavity by the second plate, does not essentially influence the flow about the first plate.

The solution of Riabouchinsky's problem is conveniently expressed in the following form:

$$\frac{dw}{v_0 dZ} = \frac{\sqrt{k^2t^2-1}}{\sqrt{t^2-1} + t\sqrt{1-k^2}} = \frac{\sqrt{t^2-1} - t\sqrt{1-k^2}}{\sqrt{k^2t^2-1}}$$

$$w = \varphi_D t, \tag{3.1}$$

where k and φ_D are constants. From (3.1) it is possible to obtain formulas for the computation of all quantities of interest to us.

The cavitation number:

$$\lambda = \frac{2\sqrt{1-k^2}}{k^2} \left(1 + \sqrt{1-k^2}\right) \tag{3.2}$$

The length of the cavity (distance between plates):

$$a = \frac{2\varphi_D}{v_0} \left[\frac{1}{k^2} E - \frac{1-k^2}{k^2} K \right] \tag{3.3}$$

The width of the cavity:

$$b = \ell + \frac{2\varphi_D}{v_0} \frac{\sqrt{1-k^2}}{1 + \sqrt{1-k^2}} \tag{3.4}$$

The length of the flat plates:

$$\ell = \frac{2\varphi_D}{v_0} \left(\frac{1-k^2}{k^2} + \frac{1}{k^2} E' - K' \right). \tag{3.5}$$

In these formulas K and E are the complete elliptic integrals of the first and second kinds with modulus k, and K' and E' are these with modulus $k' = \sqrt{1-k^2}$.

The drag coefficient of a plate:

$$C_x = 2 \left[1 - 2 \frac{\varphi_D}{v_0 \ell} \frac{1-k^2}{k^2}\right] (\lambda + 1) \tag{3.6}$$

The results of computation from formulas (3.2) to (3.6) are given on Figures 11, 12, 13, 14. On Figure 12 there is also given a curve for a/ℓ against λ computed for the scheme under consideration by an approximate formula of Weinig [7]:

$$(3.7) \qquad a/\ell = 2/\lambda$$

The discrepancy between the exact and the approximate formulas is marked.

4. The scheme of D. A. Efros

On Figure 9 is presented the scheme due to D. A. Efros 8 . The pressure along the streamlines bounding the cavity is constant. The jet flowing into the cavity is subsequently removed onto a Riemann surface. The solution of the problem is given by the formula [9]:

$$(4.1) \qquad \frac{dw}{d\zeta} = Lv_0 \frac{(\zeta^4-1)(\zeta^2+h^2)(h^2\zeta^2+1)}{\zeta(\zeta^2+d^2)^2(d^2\zeta^2+1)^2}$$

$$(4.2) \qquad \frac{dw}{v_0 dZ} = \frac{(\zeta-ih)(h\zeta-i)(\zeta-i)}{(\zeta+ih)(h\zeta+i)(\zeta+i)}$$

where ζ is a parameter with values in the semi-circle of radius 1 (Figure 10) and h, d, and L are constants. In addition h and d are connected by the condition of single-valuedness of the function $\int \frac{dZ}{d\zeta} \, d\zeta$

$$(4.3) \qquad h = \frac{1}{D/2 + \sqrt{D^2/4 - 1}}$$

where $D = (d + \frac{1}{d})^2 - (d + \frac{1}{d}) - 4.$

On Figures 11, 12, 13, and 14 are plotted the values of the dimensions of the cavity and the drag [9].

By the length of the cavity is understood the distance between the plate and the line parallel to it and tangent to the bounding streamlines of the cavity.

5. Conclusions

From an examination of Figure 11 it follows that all the theories outlined above give for C_x values very close to each other. Apparently, the determining influences on the amount of drag of a plate show themselves as the existence of discontinuous flow and the relation between the pressure at infinity and in the cavity, but not the flow pattern at the end of the cavity. The schemes of Riabouchinsky and Efros give according to Figures 12, 13, and 14 very similar results also for the dimensions of the cavity. In one of the schemes described above, the modified scheme of Betz, the cavity stretches out to infinity; in another, Riabouchinsky's scheme, a restraining body is placed at the end of the cavity; in the third scheme, that of Efros, a jet flows into the cavity proceeding thereafter onto another Riemann surface.

The introduction into the schemes of such sorts of artificial elements is done because otherwise, in the light of D'Alembert's paradox, it would be impossible to construct a stationary flow with an effective force on the plate. It is possible to propose still other schemes, modelling the free-boundary, cavitating flow about a flat plate; however, there is no foundation for supposing that they would give essentially new results.

The theory of Betz, as is known, agrees well with experiments for small cavitation numbers. With increase in the cavitation number there comes a moment when the experimental curves of the coefficients of drag and lift of a wing (with given angle of attack α), plotted against λ, turn downwards. This fact, apparently, is explained by the flow ceasing to be a free boundary flow at this point.

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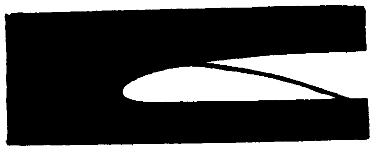

Figure 1

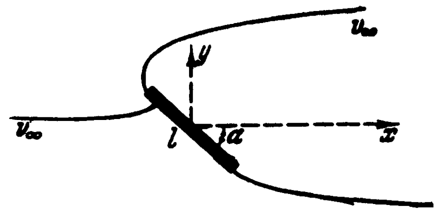

Figure 2

Figure 4

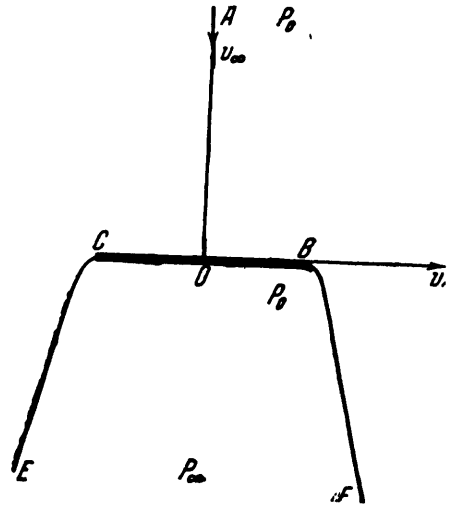

Figure 3

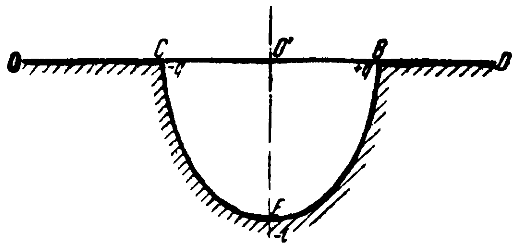

Figure 5

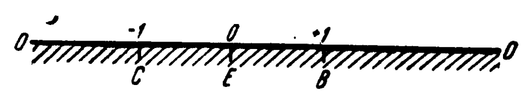

Figure 6

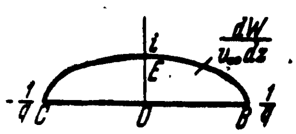

Figure 7

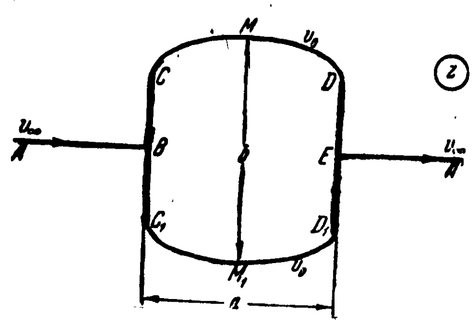

Figure 8

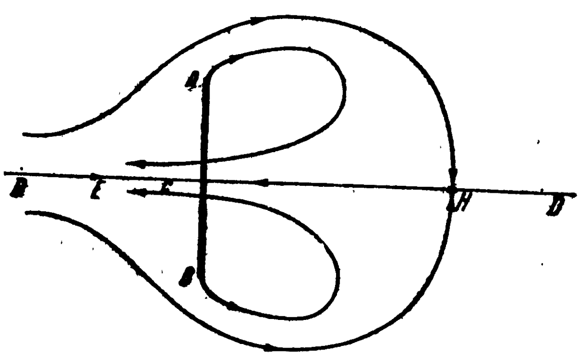

Figure 9

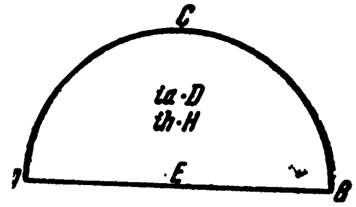

Figure 10

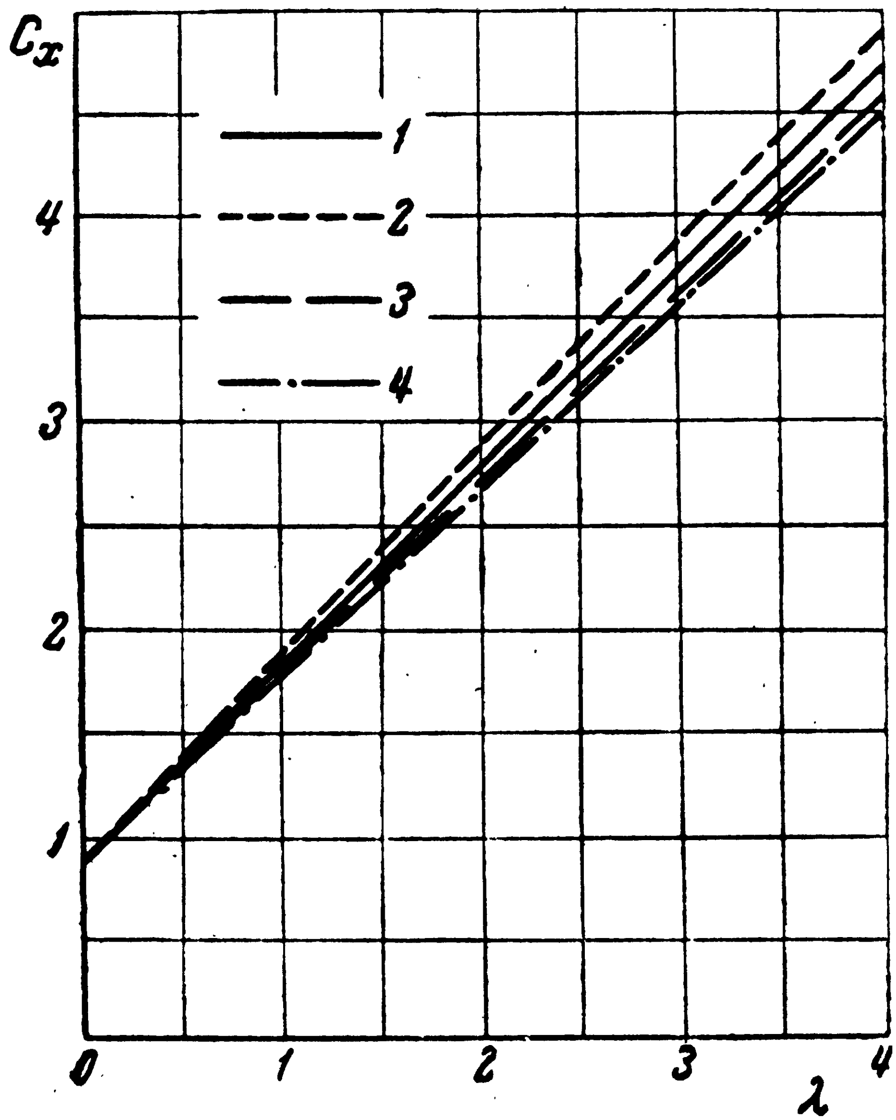

Figure 11. DEPENDENCE OF DRAG COEFFICIENT UPON CAVITATION NUMBER.

1. According to the scheme with a variable velocity on the free boundary.
2. Scheme of Betz.
3. Scheme of Efros.
4. Scheme of Riabouchinsky.

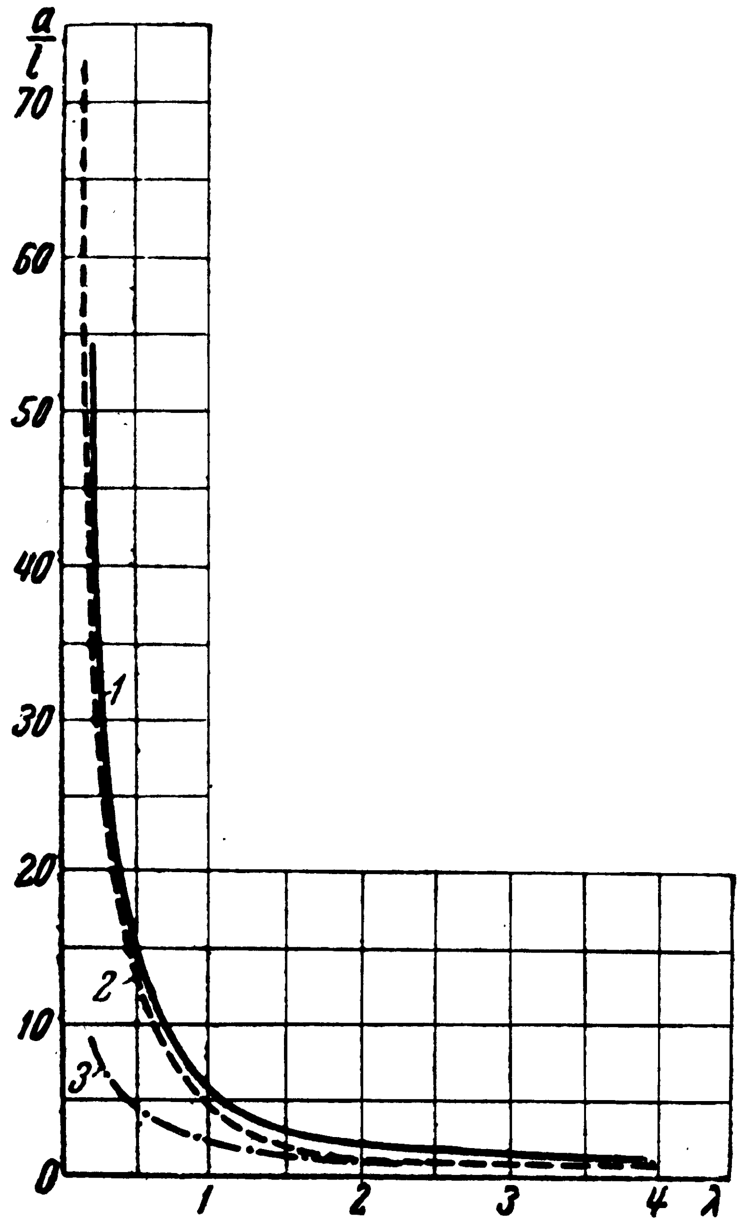

Figure 12. RELATIVE LENGTH OF THE CAVITY

1. Scheme of Efros.
2. Scheme of Riabouchinsky.
3. Approximate formula of Weinig.

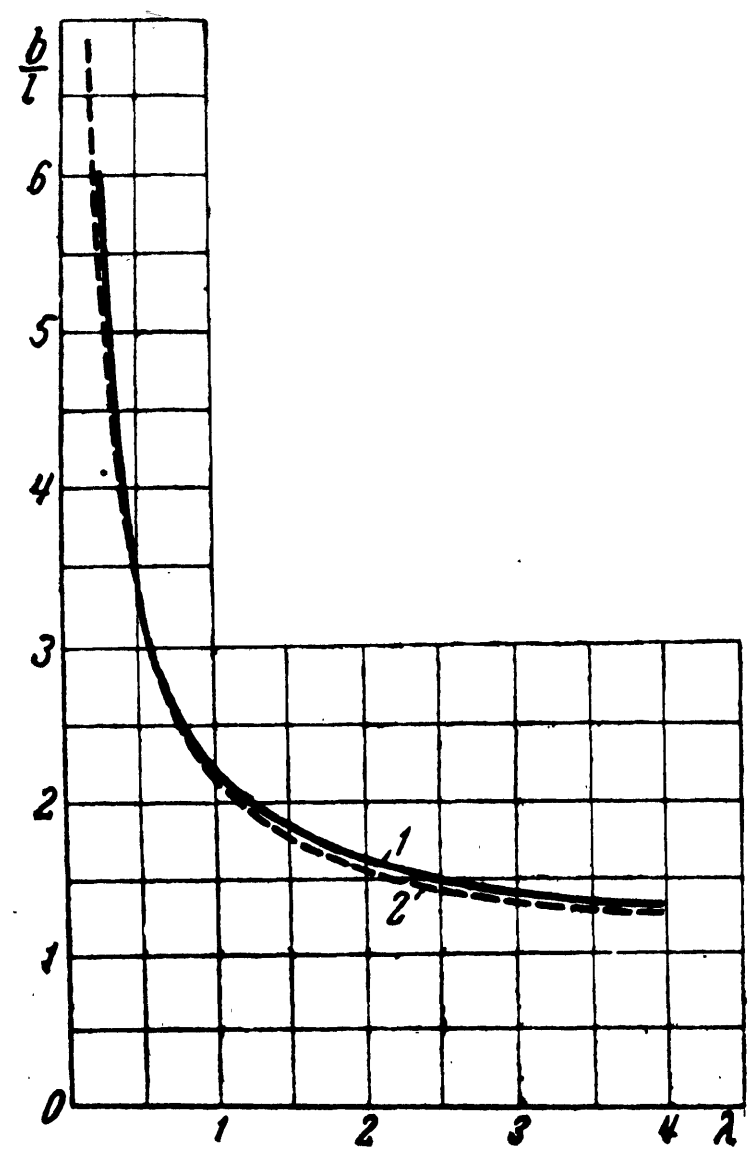

Figure 13. RELATIVE BREADTH OF CAVITY

1. Scheme of Efros.
2. Scheme of Riabouchinsky.

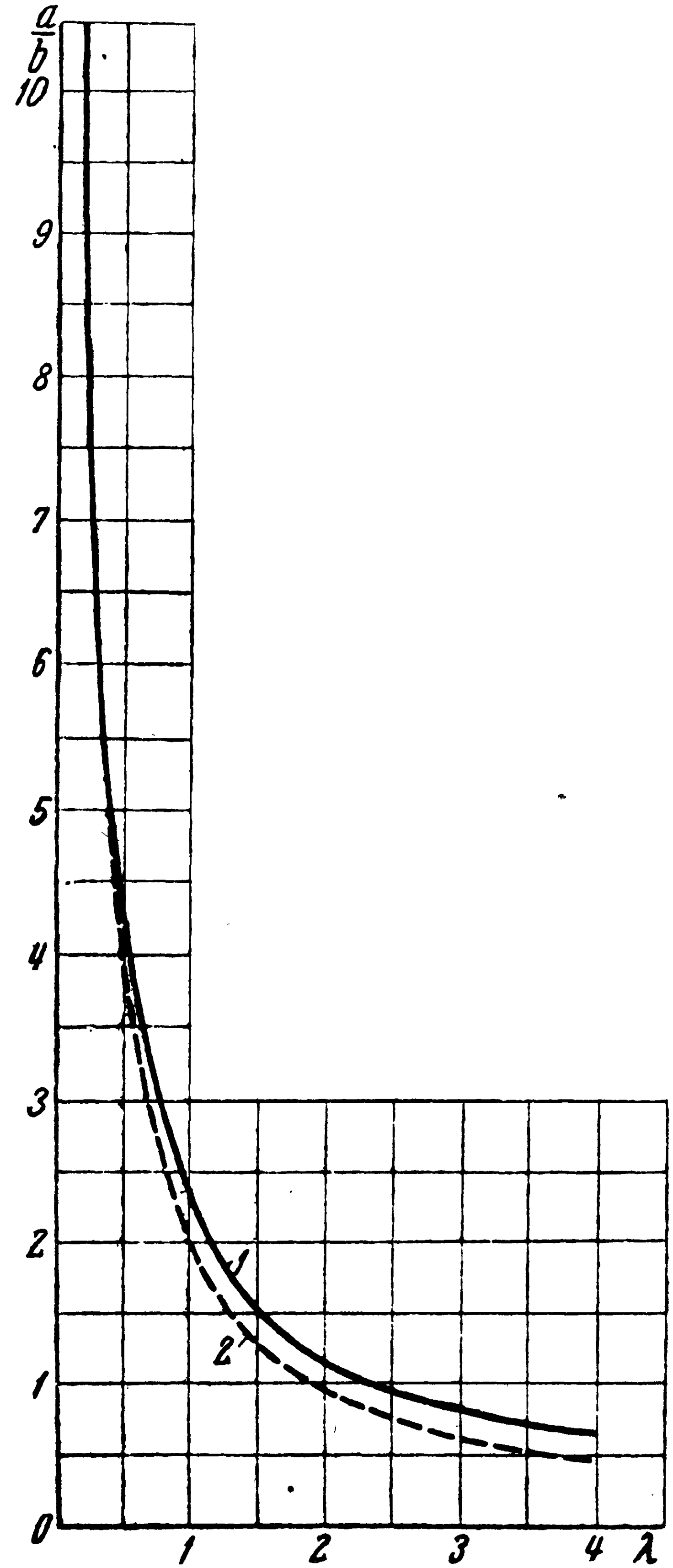

Figure 14. RATIO OF LENGTH TO BREADTH OF CAVITY

1. Scheme of Efros.
2. Scheme of Riabouchinsky.

SOME REMARKS ON GUREVICH'S PAPER

by

J. V. Wehausen

In connection with this paper it is of interest to note that similar computations have been performed in at least two other papers. Zoller (Reference 1) computed the length and diameter of the cavity behind a flat plate as well as the drag coefficient, all as a function of the cavitation number, according to the Riabouchinsky model. Gilbarg and Rock (Reference 2) performed the identical computations for this model and, in addition, for the Efros model (Wagner model in their report) computed the drag coefficient, length of the cavity, and asymptotic diameter of the reentrant jet.

In the case of the Riabouchinsky model there is practically complete agreement between the three papers. This may be seen in comparing Figures 1, 2, and 3 following these remarks with Figures 11, 12, and 13 of Gurevich's paper.

For the Wagner-Efros model Zoller made no computations. For the drag coefficient curves Gurevich and Gilbarg and Rock seem to be in good agreement as far as one can tell from the small scale of Gurevich's curves. There is no comparable data for diameters since Gurevich computed the diameter of the cavity and Gilbarg and Rock that of the reentrant jet. With regard to length of the cavity there is a difference in definition. Gurevich apparently computes the distance from the plate to the parallel tangent to the region of the cavity whereas Gilbarg and Rock compute the distance between the plate and the rear stagnation point (the distance CH on Gurevich's Figure 9). Consequently, the length computed by Gilbarg and Rock should be greater than that computed by Gurevich. On the other hand the computed points of the former show practically no difference between the two models for cavitation numbers between 0.2 and 1.0 (see Figure 2 following) whereas Gurevich's Figure 12 shows the curve for the Wagner-Efros model lying noticeably above that for the Riabouchinsky model.

Zoller gives the following approximate formulas for the length, ℓ/b, and the diameter, d/b:

$$\frac{d}{b} = \frac{4}{\pi+4}\left\{\frac{1}{\delta} + \frac{\pi}{4}\right\}$$

$$\frac{\ell}{b} = \frac{4}{\pi+4}\left\{\frac{1}{\delta^2} + \frac{3}{4} + \frac{1}{2}\ln\frac{\delta}{4}\right\}$$

where $\delta = \sigma/(2+\sigma)$. The formula for d/b is accurate within 0.6% for $\sigma \leq 3$. The formula for ℓ/b loses accuracy more rapidly.

At $\sigma = 1$, it is 15% too high, but at $\sigma = 0.5$ only 4% too high; at $\sigma = 0.4$ it is correct within a fraction of a per cent. For low cavitation numbers Gilbarg and Rock give the following approximate formula for the length ℓ/b:

$$\frac{\ell}{b} = 3.5\ \sigma^{-1.85}$$

That this formula is not as accurate as the Zoller approximation for cavitation numbers less than 1 is indicated by the following table

$\sigma \backslash \ell/b$	Exact	Zoller	G-R
1	4.186	4.77	3.5
.5	13.036	13.58	12.62
.2	67.031	67.2	68.7

Both Zoller and Gilbarg-Rock give the approximate formula

$$C_D = \frac{2\pi}{\pi + 4}\ (1 + \sigma).$$

This underestimates by less than 0.8% at $\sigma = 1$ and improves as σ decreases.

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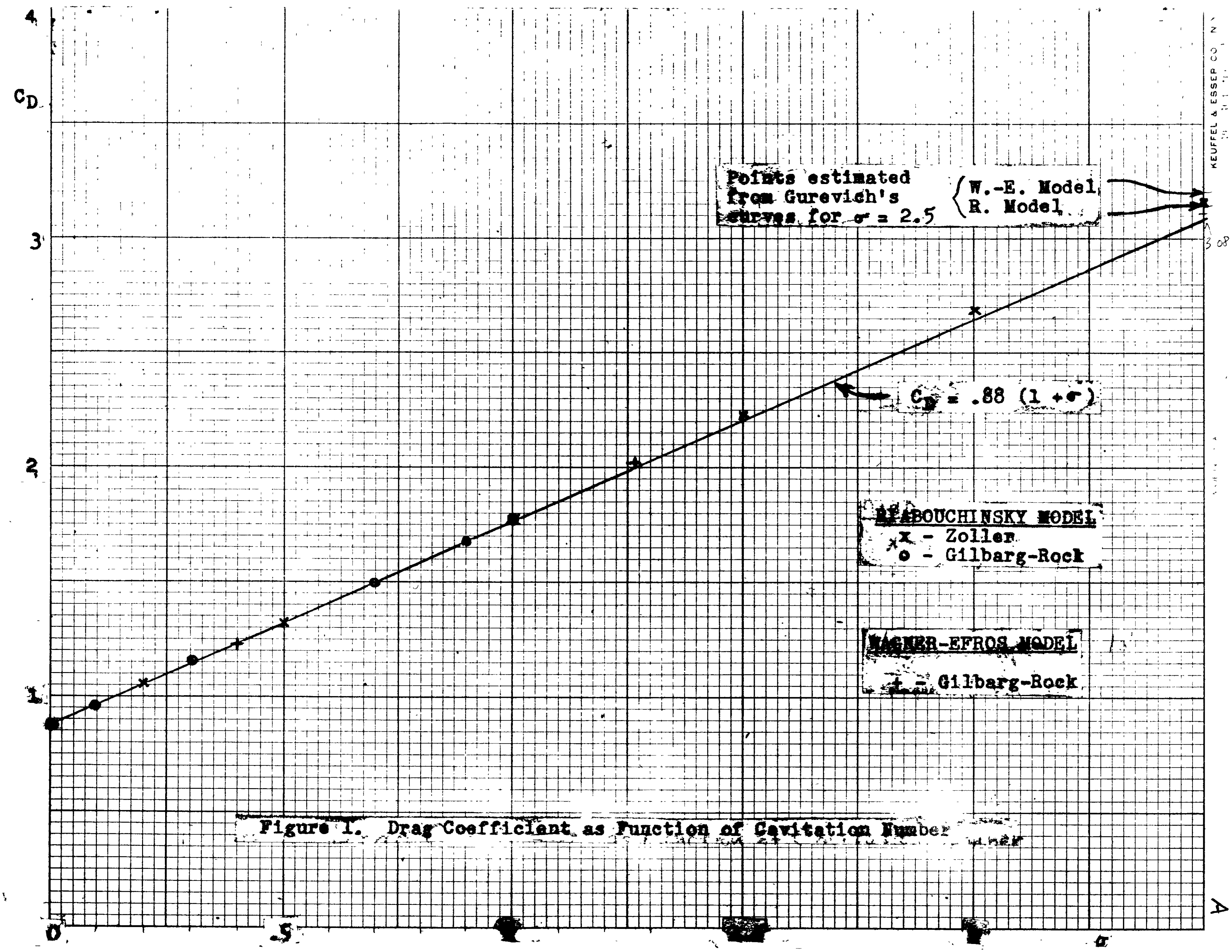

Figure I. Drag Coefficient as Function of Cavitation Number

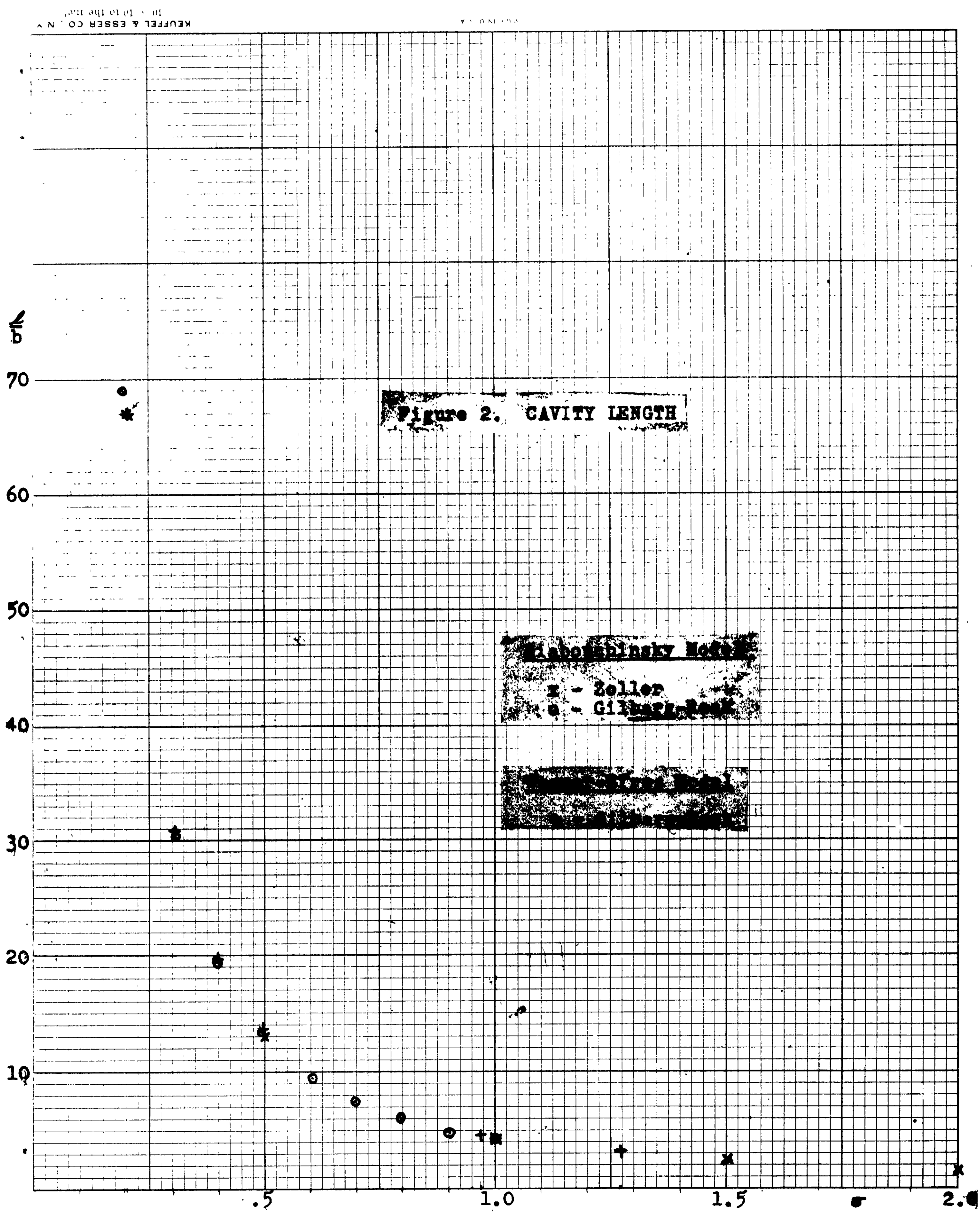

Figure 2. CAVITY LENGTH

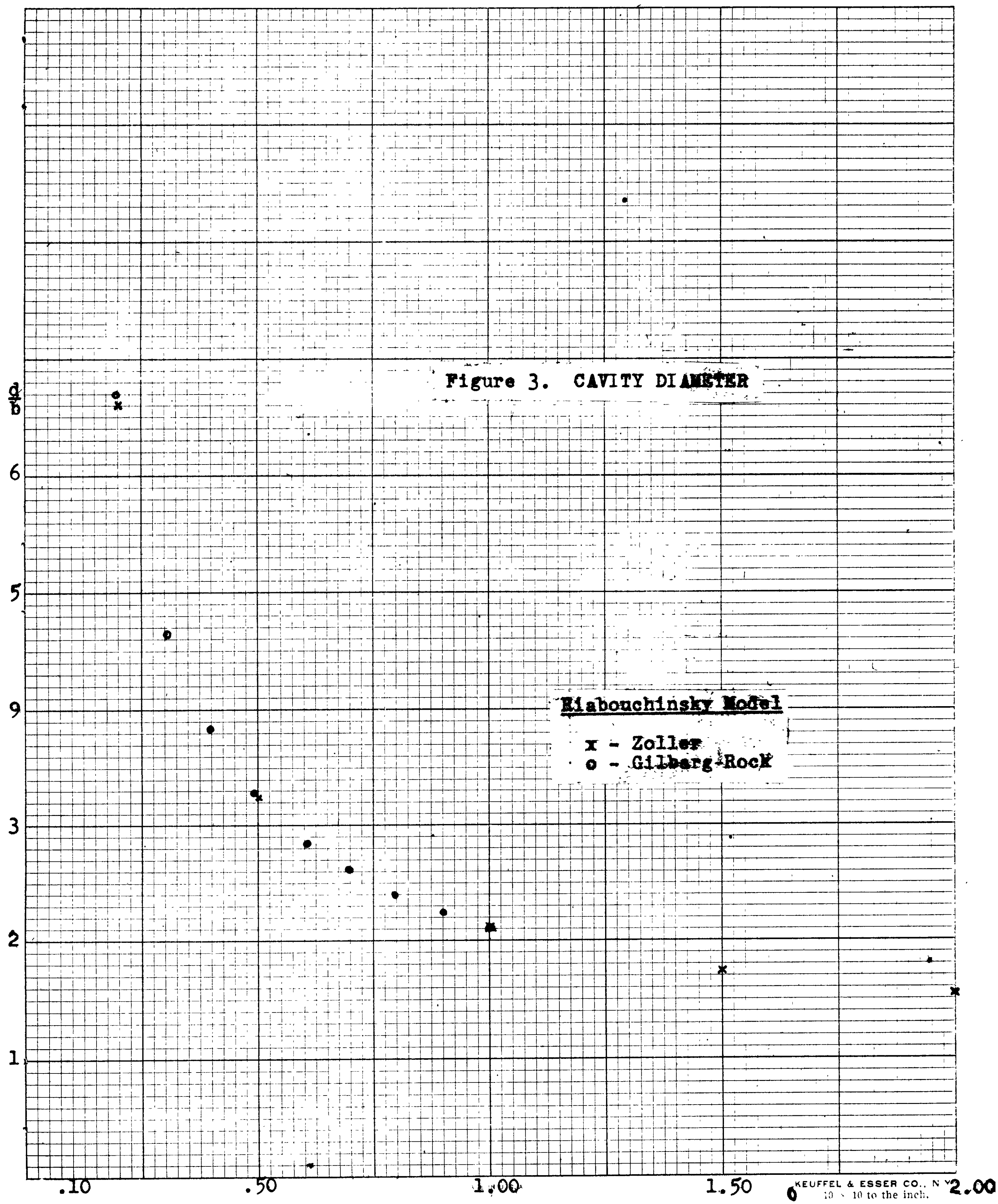

Figure 3. CAVITY DIAMETER
Riabouchinsky Model
x - Zoller
o - Gilbarg-Rock
d/b
6
5
9
3
2
1
.10
.50
1.00
1.50
2.00
KEUFFEL & ESSER CO., N.Y.
10 x 10 to the inch.