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THE ELASTIC STRESSES IN THIN-WALLED TUBES CAUSED BY INTERNAL PRESSURES CREATED BY EXPLOSIONS

BY KARL KLOTTER AND RUTH PICH

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THE ELASTIC STRESSES IN THIN-WALLED TUBES CAUSED BY INTERNAL PRESSURES CREATED BY EXPLOSIONS

(DIE ELASTISCHEN SPANNUNGEN IN DÜNNWANDIGEN ROHREN UNTER EXPLOSIONSARTIGEM INNENDRUCK)

by

Karl Klotter and Ruth Pich

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ABSTRACT

The problem concerning the precise relationship between the rise, with respect to time, of an internal pressure caused by an explosion in a tube and the tangential stress in the tube--which was taken up in earlier investigations--is raised again, and by revision of previous conceptions is solved in a manner sufficiently exact for all practical purposes. The maximum value of the tangential stress may exceed the static value which corresponds to the maximum value of the pressure, but it cannot rise by more than approximately 8/3 times the static value.

1. INTRODUCTION AND STATEMENT OF OBJECTIVES

The investigations presented here were prompted by two reports by H. Schlechtweg and R. Moufang* from the research institutes of the cast steel works of Friedrich Krupp (Essen) dated 18 November 1941 and 26 February 1942 (designated briefly as report I and report II hereafter). In report I, the authors develop approximation formulae for the elastic displacements and stresses in a thin-walled tube subjected to internal pressure varying with respect to time; in doing so, the pressure-time curve p(t) is being assumed in a form which can be adapted to the pressure curve resulting from the firing of guns.

By a numerical example in which the internal pressure is assumed in the form of

$$p(t) = P\cdot(e^{-\alpha t} - e^{-2\alpha t}) \quad \text{with } \alpha > 0 \qquad [1.1]$$

(Figure 1), it is being shown that the maximum value of the tangential stress σ_ϕ to which the tube is subjected may exceed the value which this stress would have under static internal pressure in the amount of

$$p_{static} = [p(t)]_{max} \qquad [1.2]$$

It is true that in the example referred to only the insignificant

*"The elastic stress distribution in thin-walled tubes under shock-like internal pressure when the outer surface and ends are not subjected to any force."

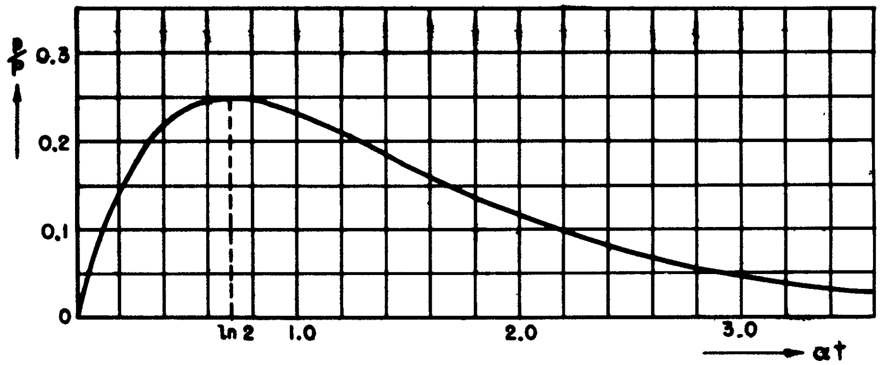

Figure 1 - Internal Pressure, $p = P\cdot(e^{-\alpha t} - e^{-2\alpha t})$

value S = 0.2 had resulted for the relatively increased stress in the center of the tube,

$$S = (\sigma_{\phi\,max} - \sigma_{\phi\,static}) \;/\; \sigma_{\phi\,static} \qquad [1.3]$$

However, it seemed desirable to know the exact relationship existing between the rise of the internal pressure p(t) and the maximum value of the tangential stress σ_ϕ. A universally valid formularization of such a law (based on the theory developed in report I) proved to be too difficult.* In report II the attempt is made to figure out, at least numerically, according to which law the maximum value of the tangential stress σ_ϕ changes with the pressure rise assuming the form [1.1] for the internal pressure p(t) in the case of a thin-walled tube of definite material and definite dimensions (such as in gun-barrels, for example). Three examples were calculated in which the parameter α corresponding to the pressure increase was ascribed the values

$$\alpha_I = 3.15\cdot10^2/\text{sec}, \quad \alpha_{II} = 3.15\cdot10^3/\text{sec}, \quad \alpha_{III} = 3.15\cdot10^4/\text{sec}$$

Using the non-dimensional quantity**

$$\bar{\alpha} = 1.11\alpha \cdot 10^{-5}\ \text{sec} \qquad [1.4]$$

we summarize the result of these numerical calculations by saying that in the investigated time interval

$$0 < \bar{\alpha} \leqq 0.35 \qquad [1.5a]$$

*This is due mainly to the complicated formulae for the free vibrations of the tube. We shall refer to this point again later on.

**The (linear) relationship between the parameter and a non-dimensional quantity designated as $\bar{\alpha}$ is expressed by varying relationships in different sections of this treatise. (Cf. [2.5] as well as [4.4a] and [5.1']. The factor $1.11 \cdot 10^{-5}$ sec in [1.4] represents a special numerical value of the factor $R_a\sqrt{\rho/\theta}$ in [4.4a] and [5.1']; in [2.5], however, the natural frequency ω (of the ring) has been used as a factor of α.)

the values

$$S(\alpha_I) = 0.02, \; S(\alpha_{II}) = 0.26, \; S(\alpha_{III}) = 2.05 \qquad [1.5b]$$

can be interpolated by the linear function

$$S(\bar{\alpha}) = 6.3\,\bar{\alpha} \qquad [1.6]$$

In a table compiled toward the end of report II (page 9) the value $\bar{\alpha}_{IV} = 10\bar{\alpha}_{III}$ (i.e. $\alpha_{IV} = 3.15 \cdot 10^5$/sec) is also quoted in addition to the parameter values $\bar{\alpha}_I$ to $\bar{\alpha}_{III}$. Although the difference $\bar{\alpha}_{IV} - \bar{\alpha}_{III}$ is nine times as great as the entire time interval 0 to $\bar{\alpha}_{III}$ on which the interpolation was based, $S(\alpha_{IV})$ is, nevertheless, calculated according to the interpolation formula [1.6] and the value thus obtained is being used in setting up the above-mentioned table.

Among other things, it is to be the objective of this investigation to examine the feasibility of such an interpolation (which yields the questionably high value of S = 22). We shall endeavor to avoid as long as possible the use of the rather complicated formulae which served as a basis for the calculations of the values [1.5b]. First, however, we need a short tabulation of the most important characteristics of the function p(t) describing the pressure curve.

If this function is given by [1.1] (Figure 1), it begins at t = 0 with the value of zero and with the slope

$$\dot{p}(0) = \alpha P \qquad [1.7a]$$

it reaches at

$$t_o = \frac{1}{\alpha} \ln 2 \qquad [1.7b]$$

a maximum in the amount of

$$p_{max} = \frac{P}{4} \qquad [1.7c]$$

and from then on, it falls asymptotically to zero. Since however, P (and thus p_{max}) is to be independent of α, the pressure rise, to be sure, becomes steeper and steeper with the rising parameter α, but at the same time the time integral of the pressure-curve

$$\int_0^\infty p(t)dt = \frac{P}{2\alpha} \qquad [1.7d]$$

(the "impulse") decreases more and more.

In connection with these circumstances we shall carry on certain considerations, the detailed proof of which is to be the main body of this paper.

As already indicated by the heading of reports I and II, the authors had undertaken to investigate the effect of a "shock-like" internal pressure upon the tube. Obviously the question "at which particular value of the parameter α determinative for the pressure rise can the curve of p(t) be labeled 'shock-like'" can not be answered at all in an absolute manner; instead it depends every time upon the structure and the material upon which the pressure is to be applied. If the structure is capable of natural (or free) vibrations, the smallest frequency of these natural vibrations offers a measure indicating how quickly the pressure must rise from zero to its maximum value in order to have a "shock-like" effect upon the structure. If the reciprocal value of the time interval $t_0 = 1/\alpha \ln 2$, which elapses from t = 0 till the maximum pressure is reached, is not even of the order of magnitude of this smallest natural frequency, we cannot yet speak of a "shock-like" curve of p(t) with regard to the structure under stress; only when $1/t_0 = \alpha/\ln 2$ becomes large as compared with the smallest natural frequency of the structure, the pressure is having a "shock-like" effect.

As soon as this requirement is met, however, the time integral of the pressure curve representing the magnitude of the shock must be considered. It is given in the preceding example by [1.7d] and decreases, therefore, according to the growing steepness of the pressure rise. What does this mean with regard to the tube under investigation? Obviously, in a time interval in which the interpolation formula [1.6] applies, the internal pressure p(t) does not yet exercise a "shock-like" effect upon the tube. Otherwise, the maximum value of σ_ϕ could not continue to increase since the force of the shock decreases continuously as α increases. Rather must we assume that the maximum values of all stresses and displacements asymptotically approach the value of zero as α increases. (The only exception to this is the radial stress σ_r, which on the inner surface of the tube balances the internal pressure p; and thus it has there the finite value $-p_{max}$ as a maximum which is independent of α.)

After these considerations, we again return to the fundamental question of our problem which is: Up to which particular value of the parameter α is an extrapolation of the linear relation [1.6] permissible beyond the time interval given by [1.5a]? Especially, does [1.6] still apply for $\alpha_{IV} = 3.15 \cdot 10^5/\text{sec}$?* With this we have reached a point, however, where we can no

*Here one might first raise the question if the pressure created by gunfire can, with the means presently at our disposal, rise to its maximum value already in the time interval $t_0(\alpha_{III})$ or even $t_0(\alpha_{IV})$, namely

$$t_0(\alpha_{III}) = \frac{\ln 2}{\alpha_{III}} = \frac{0.693}{0.315} \cdot 10^{-5}\text{sec} = 2.2 \cdot 10^{-5}\text{sec}, \quad t_0(\alpha_{IV}) = \frac{\ln 2}{\alpha_{IV}} = \frac{0.693}{0.315} \cdot 10^{-6}\text{sec} = 2.2 \cdot 10^{-6}\text{sec}$$

Since this is not the case, the investigation of the precise relation between S and α beyond the time interval [1.5a] has only theoretical significance for the time being; yet it is of fundamental importance.

longer proceed without a theoretical elasticity calculation--for we must now find out the smallest natural frequency ω_0 of the tube for which the values [1.5b] have been calculated. It can be shown (cf. Sections 4 and 5) that even the parameter value $\alpha_{III} = 3.15 \cdot 10^4$/sec reaches the order of magnitude of this smallest natural frequency. The fact that with $\alpha \approx \omega_0$ the linear relation [1.6] still applies does not contradict the above considerations. On the other hand, according to these very considerations, it seems exceedingly doubtful that the linear relation [1.6] between S and $\bar{\alpha}$ still exists also with $\alpha \approx 10\omega_0$. Rather, it is to be assumed that the value $S(\bar{\alpha}_{III}) = 2.2$ already comes very close to the upper limit given for S.

In order to be able to answer the questions suggested here as far as possible, we have once more opened up the entire problem under discussion and accordingly, in Sections 3 and 4, we have once more calculated (partly according to a somewhat modified method) the formulae developed in report I. In Section 5, the results are applied to the tube investigated in report II.

The formulae developed in Sections 3 and 4 can be greatly simplified by assuming the length of the tube to be extremely small, i.e., by considering the ring.* It will become obvious that the essential factors of our problem continue to exist even in this extreme case. The discussion of the latter will, therefore, be taken up first in the following investigations (Section 2).

2. THE RING

If in a tube, e.g. a hollow cylinder, made of elastic material not only the wall-thickness, but also the length is very small compared with the (mean) radius R (i.e., the average between the inner and outer radius), we are dealing with a ring; with respect to its radial-symmetric vibrations it represents a system of only one degree of freedom.

If the internal pressure p(t) is acting upon the ring, the differential equation for the radial displacement u(t) [whenever damping forces are not to be taken into account] reads as follows:

$$\ddot{u} + \omega^2 u = \frac{1}{\mu_0} p(t) \quad \text{with} \quad \omega^2 = \frac{EF}{\mu_0 R^2} \qquad [2.1]$$

Here E stands for the modulus of elasticity, R for the (mean) radius of the ring, F for the cross-sectional area obtained by splitting the ring, and finally μ_0 for the mass of the ring per unit of length. (The designation μ will be reserved for one of the two Lame constants which occur in Section 3.)

*This possibility has been pointed out to us by Mr. W. Flügge.

If p(t) is given by [1.1], one obtains as the particular integral of [2.1] the expression

$$u_p = \frac{P}{\mu_o} \cdot \left(\frac{e^{-\alpha t}}{\omega^2 + \alpha^2} - \frac{e^{-2\alpha t}}{\omega^2 + 4\alpha^2}\right) \qquad [2.2a]$$

representing the actual forced movement. The free movement is a harmonic vibration; it can be reduced to the form

$$u_f = \hat{a} \cos \omega t + \hat{\hat{a}} \sin \omega t \qquad [2.2b]$$

The integration constants $\hat{a}$ and $\hat{\hat{a}}$ are determined by the initial conditions for u (= $u_p + u_f$); these are

$$u(0) = 0, \ \dot{u}(0) = 0 \qquad [2.3]$$

Accordingly, we obtain for the amplitude $A = \sqrt{\hat{a}^2 + \hat{\hat{a}}^2}$ of the free vibration the expression

$$A = \sqrt{u_p^2(0) + \frac{1}{\omega^2} \dot{u}_p^2(0)} \qquad [2.4]$$

After a simple calculation--if one still introduces the (non-dimensional) quantity*

$$\bar{\alpha} = \alpha/\omega \qquad [2.5]$$

there follows for $u_p(0)$ and $\dot{u}_p(0)$ from [2.2a]

$$u_p(0) = \frac{P}{\mu_o \omega^2} \cdot \frac{3\bar{\alpha}^2}{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4} \qquad [2.6a]$$

and

$$\dot{u}_p(0) = \frac{\bar{\alpha} P}{\mu_o \omega} \cdot \frac{1 - 2\bar{\alpha}^2}{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4} \qquad [2.6b]$$

If one inserts these expressions in [2.4], one obtains finally

$$A = \frac{P}{\mu_o \omega^2} \cdot \frac{\bar{\alpha}}{\sqrt{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4}} \qquad [2.7]$$

The tangential stress σ_ϕ in the ring is proportional to the radial displacement u; i.e.

$$\sigma_\phi(t) = \frac{E}{R} u(t) \qquad [2.8]$$

For the sake of simplicity we shall, in the following section, at first speak only of the radial displacement itself.

*Cf. last footnote on page 2.

The maximum value of u(t) (and also the time of its occurrence) cannot be indicated in closed form as a glance at [2.2a] and [2.2b] will show. The value u_{max}, however, certainly does not exceed the limit

$$G = A + u_{p\,max} \qquad [2.9]$$

The (actual) forced radial displacement $u_p(t)$, which is given by [2.2a], can be written in the form

$$u_p = U_1 e^{-\alpha t} - U_2 e^{-2\alpha t} \qquad [2.10a]$$

Here

$$U_1 = \frac{P/(\mu_0\omega^2)}{1 + \bar{\alpha}^2} \;;\; U_2 = \frac{P/(\mu_0\omega^2)}{1 + 4\bar{\alpha}^2} \qquad [2.10b]$$

The maximum of $u_p(t)$ is reached at the time

$$\hat{t}_0 = \frac{1}{\alpha}\ln(2U_2/U_1) \qquad [2.11]$$

and has the value

$$u_{p\,max} = \frac{1}{4}\,U_1^2/U_2 \qquad [2.12]$$

For $t \geqq 0$, $u_{p\,max}$ represents the maximum value of $u_p(t)$ only as long as $\hat{t}_0 \geqq 0$, i.e. $2U_2 \geqq U_1$; on the other hand, if $2U_2 < U_1$, then for $t \geqq 0$ the maximum value of $u_p(t)$ is indicated by

$$u_p(0) = U_1 - U_2 \qquad [2.13]$$

From [2.12] and [2.13], and with [2.10b] it follows that

$$u_{p\,max} = \frac{P}{4\mu_0\omega^2} \cdot \frac{1 + 4\bar{\alpha}^2}{(1 + \bar{\alpha}^2)^2} \qquad [2.14a]$$

and

$$u_p(0) = \frac{P}{4\mu_0\omega^2} \cdot \frac{12\bar{\alpha}^2}{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4} \qquad [2.14b]$$

(cf. [2.6a]). According to whether $\bar{\alpha} < \frac{1}{2}\sqrt{2}$ $(2U_2 > U_1)$ or $\bar{\alpha} > \frac{1}{2}\sqrt{2}$ $(2U_2 < U_1)$, either

$$G = G_a = \frac{P}{4\mu_0\omega^2}\left[\frac{4\bar{\alpha}}{\sqrt{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4}} + \frac{1 + 4\bar{\alpha}^2}{(1 + \bar{\alpha}^2)^2}\right] \qquad [2.9a]$$

or

$$G = G_b = \frac{P}{4\mu_0\omega^2}\left[\frac{4\bar{\alpha}}{\sqrt{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4}} + \frac{12\bar{\alpha}^2}{1 + 5\bar{\alpha}^2 + 4\bar{\alpha}^4}\right] \qquad [2.9b]$$

represents the upper limit for the maximum value of the radial displacement u(t).

For $\bar{\alpha} = \frac{1}{2}\sqrt{2}$ $G_a = G_b = \frac{8}{3} \cdot P/(4\mu_o\omega^2)$.

As we can see from [2.1], the factor $P/(4\mu_o\omega^2)$ in [2.9a] or [2.9b] represents the value of the radial displacement u in case a static internal pressure in the amount of

$$p_{static} = [p(t)]_{max}, \text{ thus } p_{static} = \frac{P}{4} \qquad [2.15]$$

acts upon the ring. (Cf. [1.2] and [1.7c].) Hence we can write

$$\frac{P}{4\mu_o\omega^2} = u_{static} \qquad [2.16]$$

The function $g(\bar{\alpha}) = G(\bar{\alpha})/u_{static}$ thus represents the upper limit for the quotient u_{max}/u_{static} and likewise (because of [2.8]) for the quotient $\sigma_{\phi\,max}/\sigma_{\phi\,static}$. This function begins at $\bar{\alpha} = 0$ with the value $g(0) = 1$ and the slope* $dg/d\bar{\alpha} = 4$; already at $\bar{\alpha} = \frac{1}{2}\sqrt{2}$ (thus $\alpha = \omega/\sqrt{2}$), $g(\bar{\alpha})$ reaches a maximum value of $g_{max} = 8/3$ and from there on it drops asymptotically to zero (Figure 2).

Hence, this course of the upper limit $g(\bar{\alpha})$ for the quotient $\sigma_{\phi\,max}/\sigma_{\phi\,static}$ corresponds absolutely to the general considerations of Section 1. (Concerning the validity of the formulae derived here beyond $\bar{\alpha} = 1$ compare, however, the observations at the end of Section 5.)

3. THE TUBE

After dealing in Section 2 with the ring as a special case of the tube, we now turn to the tube of finite length and also of (for the time being) finite wall-thickness. Here also we presuppose that displacements other than radial-symmetric ones need not be considered.

---

*
$$\frac{dg}{d\bar{\alpha}} = \frac{da}{d\bar{\alpha}} + \frac{4\bar{\alpha}}{(1+\bar{\alpha}^2)^3}(1 - 2\bar{\alpha}^2) \quad \text{for } \bar{\alpha} \leq \frac{1}{2}\sqrt{2}$$

$$\frac{dg}{d\bar{\alpha}} = \frac{da}{d\bar{\alpha}} + 24\bar{\alpha} \cdot \frac{1+2\bar{\alpha}^2}{(1+5\bar{\alpha}^2+4\bar{\alpha}^4)^2}(1 - 2\bar{\alpha}^2) \quad \text{for } \bar{\alpha} \geq \frac{1}{2}\sqrt{2}$$

herein

$$\frac{da}{d\bar{\alpha}} = 4 \cdot \frac{1+2\bar{\alpha}^2}{(1+5\bar{\alpha}^2+4\bar{\alpha}^4)^{\frac{3}{2}}}(1 - 2\bar{\alpha}^2)$$

so that the following holds true:

$$\frac{dg}{d\bar{\alpha}} \gtreqless 0 \quad \text{for } \bar{\alpha} \lesseqgtr \frac{1}{2}\sqrt{2}$$

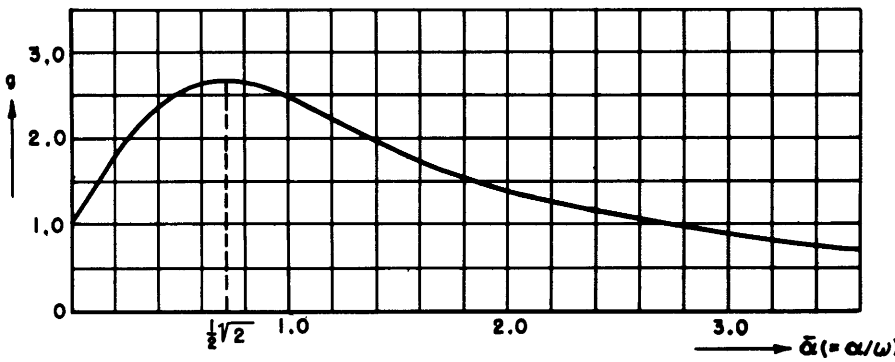

Figure 2 - Upper Limit $g(\bar{\alpha})$ for the Quotient $[\sigma_\phi(t)]_{max}/\sigma_{\phi\,static}$ for the Ring (σ_ϕ: Tangential Stress)

For the radial displacement $u = u(r,z,t)$ and the axial displacement $w = w(r,z,t)$ we can apply the differential equations

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1-M}{2}\frac{\partial^2 u}{\partial z^2} + \frac{1+M}{2}\frac{\partial^2 w}{\partial r \partial z} = \frac{\rho}{\theta} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1+M}{2}\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + \frac{1-M}{2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2} = \frac{\rho}{\theta}\frac{\partial^2 w}{\partial t^2} \qquad [3.1]$$

and the initial conditions

$$u(r,z,0) \equiv 0, \quad \left[\frac{\partial u}{\partial t}\right]_{t=0} \equiv 0$$

$$w(r,z,0) \equiv 0, \quad \left[\frac{\partial w}{\partial t}\right]_{t=0} \equiv 0 \qquad [3.2]$$

In [3.1] ρ denotes the density of the material; furthermore,

$$\theta = \frac{1+M}{(1-M)(1+2M)} E, \quad M = \frac{1}{m-1}$$

where m designates Poisson's ratio and E the modulus of elasticity.

The tangential stress, radial stress, axial stress, and the shear stress are given by

$$\sigma_\phi = \theta \cdot \left(\frac{u}{r} + M\frac{\partial u}{\partial r} + M\frac{\partial w}{\partial z}\right)$$

$$\sigma_r = \theta \cdot \left(M\frac{u}{r} + \frac{\partial u}{\partial r} + M\frac{\partial w}{\partial z}\right) \qquad [3.3]$$

$$\sigma_z = \theta \cdot \left(M\frac{u}{r} + M\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) .$$

$$\tau_{rz} = \mu \cdot \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \quad \text{with } \mu = \frac{1+M}{2(1+2M)} E$$

If we assume that besides the internal pressure p(t) no forces act upon the tube, then there apply upon the surface of the tube for the radial stress σ_r or for the shear stress τ_{rz} the following boundary conditions:

$$\begin{aligned} \sigma_r &\equiv -p(t) \quad \text{for} \quad r = R_i \\ \sigma_r &\equiv 0 \quad \text{for} \quad r = R_a \\ \tau_{rz} &\equiv 0 \quad \text{for} \quad r = R_i \text{ and } r = R_a \end{aligned} \qquad [3.4a]$$

Furthermore, there apply at the tube ends for the axial stress σ_z and the shear stress τ_{rz} the boundary conditions

$$\sigma_z \equiv 0 \text{ and } \tau_{rz} \equiv 0 \quad \text{for} \quad z = \pm L \qquad [3.4b]$$

Herein R_i designates the inner radius, R_a the outer radius, and L one-half of the length of the tube.

Here we do not yet represent the internal pressure p(t) in the form [1.1]; instead we first choose a more general representation which is more suitable for the following calculations, namely

$$p(t) = \sum_{n=0}^{N} P_n e^{-\alpha_n t} \quad \text{with} \quad \alpha_n \geqq 0 \quad \text{and} \quad \sum_{n=0}^{N} P_n = 0 \qquad [3.5]$$

The form of the differential equations [3.1] and those of the pressure curve p(t) in connection with the boundary conditions [3.4] suggests the following approximate solutions for the forced and for the free displacements u and w:

$$\begin{aligned} u^{(p)} &= \sum_{n=0}^{N} u_n^{(p)} \quad \text{with } u_n = \left[U_n(r) + \hat{U}_n(r)\cosh(\beta_n z)\right]e^{-\alpha_n t} \\ w^{(p)} &= \sum_{n=0}^{N} w_n^{(p)} \quad \text{with } w_n = \hat{W}_n(r)\sinh(\beta_n z)e^{-\alpha_n t} \end{aligned} \qquad [3.6a]$$

or*

$$\begin{aligned} u^{(f)} &= \sum_{k=0}^{\infty} u_k^{(f)} \quad \text{with} \quad u_k = \hat{A}_k(r)\cos(\gamma_k z)\cos(\omega_k t) \\ w^{(f)} &= \sum_{k=0}^{\infty} w_k^{(f)} \quad \text{with} \quad w_k = \hat{C}_k(r)\sin(\gamma_k z)\cos(\omega_k t) \end{aligned} \qquad [3.6b]$$

---

*For the sake of simplicity the approximate solutions for the free displacements are indicated in an incomplete manner only; the expressions $u_k^{(f)}$ and $w_k^{(f)}$ in [3.6b] are to be supplemented in the following manner:

(Continued on the next page.)

From the foregoing, according to [3.3], we obtain the following approximate solutions for the stresses:

$$\sigma_{\phi,n}^{(p)} = \theta \cdot \left[\frac{1}{r} U_n + MU_n' + \left(\frac{1}{r} \hat{U}_n + M\hat{U}_n' + M\beta_n\hat{W}_n\right)\cosh(\beta_n z)\right] e^{-\alpha_n t}$$

$$\sigma_{r,n}^{(p)} = \theta \cdot \left[\frac{M}{r} U_n + U_n' + \left(\frac{M}{r} \hat{U}_n + \hat{U}_n' + M\beta_n\hat{W}_n\right)\cosh(\beta_n z)\right] e^{-\alpha_n t} \qquad [3.7a]$$

$$\sigma_{z,n}^{(p)} = \theta \cdot \left[M \cdot \left(\frac{1}{r} U_n + U_n'\right) + \left[M \cdot \left(\frac{1}{r} \hat{U}_n + \hat{U}_n'\right) + \beta_n\hat{W}_n\right] \cosh(\beta_n z)\right] e^{-\alpha_n t}$$

$$\tau_{rz,n}^{(p)} = \mu(\beta_n\hat{U}_n + \hat{W}_n') \cdot \sinh(\beta_n z) e^{-\alpha_n t}$$

or

$$\sigma_{\phi,k}^{(f)} = \theta \cdot \left[\frac{1}{r} \hat{A}_k + M\hat{A}_k' + M\gamma_k\hat{C}_k\right]\cos(\gamma_k z)\cos(\omega_k t)$$

$$\sigma_{r,k}^{(f)} = \theta \cdot \left[\frac{M}{r} \hat{A}_k + \hat{A}_k' + M\gamma_k\hat{C}_k\right]\cos(\gamma_k z)\cos(\omega_k t)$$

$$\sigma_{z,k}^{(f)} = \theta \cdot \left[M \cdot \left(\frac{1}{r} \hat{A}_k + \hat{A}_k'\right) + \gamma_k\hat{C}_k\right]\cos(\gamma_k z)\cos(\omega_k t) \qquad [3.7b]$$

$$\tau_{rz,k}^{(f)} = \mu \cdot (-\gamma_k\hat{A}_k + \hat{C}_k')\sin(\gamma_k z)\cos(\omega_k t)$$

The functions of r, which occur in the approximate solutions [3.6a] or [3.6b], thus $U_n(r)$, $\hat{U}_n(r)$, and $\hat{W}_n(r)$ or $\hat{A}_k(r)$ and $\hat{C}_k(r)$, must, because of **the equilibrium conditions [3.1], satisfy the following differential equations:**

$$\left(\alpha_n^2 \frac{\rho}{\theta} + \frac{1}{r^2}\right) U_n - \left(\frac{1}{r} U_n' + U_n''\right) = 0 \qquad [3.8a]$$

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*(Continued from the previous page.)

$$u_k^{(f)} = [\hat{A}_k(r)\cos(\omega_k t) + \hat{\hat{A}}_k(r)\sin(\omega_k t)]\cos(\gamma_k z)$$

$$w_k^{(f)} = [\hat{C}_k(r)\cos(\omega_k t) + \hat{\hat{C}}_k(r)\sin(\omega_k t)]\sin(\gamma_k z)$$

The equations derived hereafter for the determination of $\hat{A}_k(r)$ or $\hat{C}_k(r)$ apply in the same form also for $\hat{\hat{A}}_k(r)$ or $\hat{\hat{C}}_k(r)$. The differences between $\hat{\hat{A}}_k(r)$ and $\hat{A}_k(r)$ or between $\hat{\hat{C}}_k(r)$ and $\hat{C}_k(r)$ become apparent only when one attempts the complete determination of all these functions; the initial conditions [3.2] serve this purpose.

$$\left(\alpha_n^2 \frac{\rho}{\theta} + \frac{1}{r^2} - \frac{1-M}{2} \beta_n^2\right)\hat{U}_n - \left(\frac{1}{r} \hat{U}_n' + \hat{U}_n''\right) - \frac{1+M}{2} \beta_n \hat{W}_n' = 0$$

$$\left(\alpha_n^2 \frac{\rho}{\theta} - \beta_n^2\right)\hat{W}_n - \frac{1-M}{2}\left(\frac{1}{r} \hat{W}_n' + \hat{W}_n''\right) - \frac{1+M}{2} \beta_n \cdot \left(\frac{1}{r} \hat{U}_n + \hat{U}_n'\right) = 0 \qquad [3.9a]$$

or

$$\left(-\omega_k^2 \frac{\rho}{\theta} + \frac{1}{r^2} + \frac{1-M}{2} \gamma_k^2\right)\hat{A}_k - \left(\frac{1}{r} \hat{A}_k' + \hat{A}_k''\right) - \frac{1+M}{2} \gamma_k \hat{C}_k' = 0$$

$$\left(-\omega_k^2 \frac{\rho}{\theta} + \gamma_k^2\right)\hat{C}_k - \frac{1-M}{2}\left(\frac{1}{r} \hat{C}_k' + \hat{C}_k''\right) + \frac{1+M}{2} \gamma_k \cdot \left(\frac{1}{r} \hat{A}_k + \hat{A}_k'\right) = 0 \qquad [3.9b]$$

These five functions of r are fixed by these five differential equations (as the complete integrals of the latter); yet there are left altogether 10 (arbitrary) integration constants. Besides these integration constants, the constants β_n or γ_k and ω_k also remain at our disposal. However, the boundary conditions [3.4] for the stresses and the initial conditions [3.2] for the displacements are now to be taken into consideration.

Because of the boundary conditions [3.4a] for σ_r and τ_{rz} applicable to the surface of the tube, the majority of these (altogether 13) available constants is fixed now by the fact that the functions of r for $r = R_i$ or $r = R_a$, which we are discussing here, must satisfy the following equations (cf. [3.7a] or [3.7b]):

$$\frac{M}{r} U_n + U_n' = -P_n/\theta \quad \text{or} \quad = 0 \quad \text{for} \quad r = R_i \quad \text{or} \quad r = R_a \qquad [3.10a]$$

$$\frac{M}{r} \hat{U}_n + \hat{U}_n' + M\beta_n \hat{W}_n = 0 \quad \text{for } r = R_i \quad \text{and} \quad r = R_a \qquad [3.11a]$$

$$\beta_n \hat{U}_n + \hat{W}_n' = 0 \quad \text{for } r = R_i \quad \text{and} \quad r = R_a \qquad [3.12a]$$

$$\frac{M}{r} \hat{A}_k + \hat{A}_k' + M\gamma_k \hat{C}_k = 0 \quad \text{for} \quad r = R_i \quad \text{and} \quad r = R_a \qquad [3.11b]$$

$$-\gamma_k \hat{A}_k + \hat{C}_k' = 0 \quad \text{for} \quad r = R_i \quad \text{and} \quad r = R_a \qquad [3.12b]$$

For the two integration constants contained in the complete integral $U_n(r)$ of the differential equation [3.8a] the two equations [3.10a] represent a non-homogeneous system of algebraic linear equations. Therefore, these two integration constants can now be fully determined from them. The complete integrals $\hat{U}_n(r)$ and $\hat{W}_n(r)$ of the (coupled) differential equations [3.9a] contain altogether four (arbitrary) integration constants. For these the four equations [3.11a] and [3.12a] represent a homogeneous system of algebraic linear equations so that only the relation of the four integration constants can be determined from them. From the disappearance of the determinant of this homogeneous system a relation results between the constants β_n and the parameter α_n. Since the latter is given with the internal pressure p(t), β_n is completely determined by the relation referred to.

The functions $\hat{A}_k(r)$ and $\hat{C}_k(r)$ determining the free vibration can be dealt with in a manner very similar to the functions $\hat{U}_n(r)$ and $\hat{W}_n(r)$. As complete integrals of the (coupled) differential equations [3.9b] they likewise contain altogether four (arbitrary) integral constants, the relation of which can be determined from the four homogeneous linear equations [3.11b] and [3.12b]. From the disappearance of the determinant of this system of equations results a relation between the two constants γ_k and ω_k. However, in contrast to the parameter α_n the constant ω_k is not given, but is only determined by the relation just referred to--for the time being, however, only as a function of the constant γ_k, which is yet to be determined.

For the complete determination of the integration constants contained in $\hat{A}_k(r)$ and $\hat{C}_k(r)$, the initial conditions [3.2] for the displacements u and w* can be used. The remaining boundary conditions must then be used for the complete determination of the integral constants contained in $\hat{U}_n(r)$ and $\hat{W}_n(r)$ as well as for the determination of the constants γ_k (or ω_k); these are the equations [3.4b] which are to be satisfied by the axial stress σ_z and the shear stress τ_{rz} at the tube ends, i.e. for $z = \pm L$. However, neither these nor the initial conditions can any longer be retained exactly without contradicting the results obtained thus far. If we first consider the boundary conditions, a glance at the equations [3.7a] and [3.7b] will show that because of [3.4b] the following conditional equations which are valid for the entire interval $R_i \leq r \leq R_a$ (not only for its boundaries), would result for the functions $U_n(r)$, $\hat{U}_n(r)$, and $\hat{W}_n(r)$ or $\hat{A}_k(r)$ and $\hat{C}_k(r)$:

$$M \cdot \left(\frac{1}{r} U_n + U_n'\right) + \left[M \cdot \left(\frac{1}{r} \hat{U}_n + \hat{U}_n'\right) + \beta_n \hat{W}_n\right] \cosh(\beta_n L) \equiv 0 \quad [3.13a]$$

$$(\beta_n \hat{U}_n + \hat{W}_n') \cdot \sinh(\beta_n L) \equiv 0 \quad [3.14a]$$

or

$$\left[M \cdot \left(\frac{1}{r} \hat{A}_k + \hat{A}_k'\right) + \gamma_k \hat{C}_k\right] \cos(\gamma_k L) \equiv 0 \quad [3.13b]$$

$$(-\gamma_k \hat{A}_k + \hat{C}_k') \cdot \sin(\gamma_k L) \equiv 0 \quad [3.14b]$$

Because of the equations [3.12a] or [3.12b] the equation [3.14a] or [3.14b] is satisfied at least at the two limits ($r = R_i$ and $r = R_a$) of its prescribed area of validity.

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*Cf. with this the annotation on pages 10 and 11 as well as the qualifying equation [4.12b] for $\bar{\omega}_k$.

The equation [3.13b] can be maintained in the entire interval $R_i \leq r \leq R_a$, if the constant γ_k, not thus far determined, is disposed of in such a manner that the following holds true:

$$\cos(\gamma_k L) = 0 \quad \text{for} \quad k = 0,1,2,\ldots\infty \qquad [3.13'b]$$

However, the result of this is that the initial conditions for u and $\partial u/\partial t$ at $z = \pm L$ (thus at the tube ends also) can no longer be satisfied.

If finally in the case of the equation [3.13a] one is content with merely satisfying it at the one limit of its prescribed area of validity--thus e.g. for $r = R_a$--then [3.13a] can be used for the complete determination of the integral constants contained in $\hat{U}_n(r)$ and $\hat{W}_n(r)$.

Also under the initial conditions [3.2] [which were to be used for the complete determination of the integration constants contained in $\hat{A}_k(r)$ and $\hat{C}_k(r)$], one must give up the idea of satisfying them in their entire area of validity. This applies with respect to the (independent) variable z only for $z = \pm L$, but with respect to the (independent) variable r we must limit ourselves to satisfying the equation [3.2] only for a particular value, e.g. $r = R_a$.

Further constants which might be used for the more exact retention of the boundary or initial conditions are no longer available to us now. But the fact that these conditions can, in every case, be satisfied at least for a definite particular value r, namely one of the limits of the interval $R_i \leq r \leq R_a$, suggests that for a tube of very slight wall-thickness, i.e., for

$$\frac{R_a - R_i}{R_a} \ll 1 \qquad [3.15]$$

the solutions attempted here are at least approximately correct.

If we, therefore, limit our further considerations to thin-walled tubes, we can then turn to the task of calculating the functions $U_n(r)$, $\hat{U}_n(r)$,...as integrals of [3.8a], [3.9a], [3.9b].

4. THE THIN-WALLED TUBE

The integration of the (ordinary) differential equations [3.8a], [3.9a], [3.9b] will succeed only if the dependent variables--$U_n(r)$, $\hat{U}_n(r)$,...--are developed in series. Since, for reasons explained in Section 3, we are forced to limit very narrowly the interval ($R_i \leq r \leq R_a$) for the independent variable according to rule [3.15], it is advantageous to develop according to powers of the (non-dimensional) variable

$$\bar{h} = 1 - \frac{r}{R_a} \qquad [4.1]$$

While r increases from R_i to R_a, $\bar{h}$ drops from the value

$$\bar{H} = \frac{R_a - R_i}{R_a} \qquad [4.1a]$$

to zero. In the case of a thin-walled tube, therefore, the variable $\bar{h}$ (even its maximum value $\bar{H}$) remains very small compared to 1. Inasmuch as our solutions [3.6] apply only approximately anyway, we are justified in breaking off the power series, after the term $\bar{h}^1$ or--under certain conditions--even after the absolute term.

In the qualifying equations for the variables $U_n(r)$, $\hat{U}_n(r)$,...derived in Section 3, there occur, besides these functions themselves, also expressions such as U_n', $\frac{1}{r} U_n$, $\hat{U}_n''$,.... For the power-series developments of these expressions general formulae can be compiled. First, we write, therefore, for $U_n(r)$ or $\hat{U}_n(r)$,...

$$V(r) = \sum_{\nu=0}^{\infty} V_\nu \bar{h}^\nu \quad \text{with} \quad \bar{h} = 1 - \frac{r}{R_a}$$

In this way, we obtain further:*

$$V' = -\frac{1}{R_a} \sum_{\nu=0}^{\infty} (\nu + 1) V_{\nu+1} \bar{h}^\nu$$

$$V'' = \frac{1}{R_a^2} \sum_{\nu=0}^{\infty} (\nu + 1)(\nu + 2) V_{\nu+2} \bar{h}^\nu$$

$$\frac{1}{r} V = \frac{1}{R_a} \sum_{\nu=0}^{\infty} V_\nu^{I} \bar{h}^\nu \quad \text{with} \quad V_\nu^{I} = V_o + V_1 + \ldots + V_\nu$$

$$\frac{1}{r^2} V = \frac{1}{R_a^2} \sum_{\nu=0}^{\infty} V_\nu^{II} \bar{h}^\nu \quad \text{with} \quad V_\nu^{II} = (\nu + 1) V_o + \nu V_1 + \ldots + V_\nu$$

$$\frac{1}{r} V' = -\frac{1}{R_a^2} \sum_{\nu=0}^{\infty} V_\nu^{III} \bar{h}^\nu \quad \text{with} \quad V_\nu^{III} = V_1 + 2V_2 + \ldots + (\nu + 1) V_{\nu+1}$$

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*The following relationships are to be taken into consideration

$$\frac{1}{r} = \frac{1}{R_a} \cdot \frac{1}{1 - \bar{h}}, \quad \text{thus} \quad \frac{1}{r} = \frac{1}{R_a} \cdot \sum_{\nu=0}^{\infty} \bar{h}^\nu$$

$$V' = \frac{dV}{dr}, \quad \text{i.e.} \quad V' = -\frac{1}{R_a} \cdot \frac{dV}{d\bar{h}}$$

Using the power-series developments here described, we obtain (through comparison of coefficients) from the differential equations [3.8a], [3.9a], [3.9b] recurrence formulae (Rekursionsformeln) for the coefficients $U_{n\nu}$, $\hat{U}_{n\nu}$, ...of these developments. By comparing the absolute terms of the series the following relationships develop between the coefficients with the subscripts $\nu = 0$, $\nu = 1$, and $\nu = 2$:

$$(\bar{\alpha}_n^2 + 1)U_{no} + U_{n1} - 2U_{n2} = 0 \qquad [4.2a]$$

$$\left(\bar{\alpha}_n^2 + 1 - \frac{1 - M}{2}\bar{\beta}_n^2\right)\hat{U}_{no} + \hat{U}_{n1} - 2\hat{U}_{n2} + \frac{1 + M}{2}\bar{\beta}_n\hat{W}_{n1} = 0$$

$$(\bar{\alpha}_n^2 - \bar{\beta}_n^2)\hat{W}_{no} + \frac{1 - M}{2}\hat{W}_{n1} - (1 - M)\hat{W}_{n2} - \frac{1 + M}{2}\bar{\beta}_n \cdot (\hat{U}_{no} - \hat{U}_{n1}) = 0 \qquad [4.3a]$$

or

$$\left(-\bar{\omega}_k^2 + 1 + \frac{1 - M}{2}\bar{\gamma}_k^2\right)\hat{A}_{ko} + \hat{A}_{k1} - 2\hat{A}_{k2} + \frac{1 + M}{2}\bar{\gamma}_k\hat{C}_{k1} = 0$$

$$(-\bar{\omega}_k^2 + \bar{\gamma}_k^2)\hat{C}_{ko} + \frac{1 - M}{2}\hat{C}_{k1} - (1 - M)\hat{C}_{k2} + \frac{1 + M}{2}\bar{\gamma}_k \cdot (\hat{A}_{ko} - \hat{A}_{k1}) = 0 \qquad [4.3b]$$

For the purpose of abbreviation the non-dimensional quantities

$$\bar{\alpha}_n = \alpha_n \cdot R_a \cdot \sqrt{\frac{\rho}{\theta}} \qquad [4.4a]$$

$$\bar{\beta}_n = \beta_n \cdot R_a \qquad [4.5a]$$

or

$$\bar{\omega}_k = \omega_k \cdot R_a \cdot \sqrt{\frac{\rho}{\theta}} \qquad [4.4b]$$

$$\bar{\gamma}_k = \gamma_k \cdot R_a \qquad [4.5b]$$

have been introduced here.

If we now apply the power-series developments also to those qualifying equations for $U_n(r)$, $\hat{U}_n(r)$, ..., which resulted from the boundary conditions [3.4a] (valid on the surface of the tube) thus to the equations [3.10a], [3.11a], and [3.12a] or [3.11b] and [3.12b], we can no longer carry out a comparison of coefficients since those equations are supposed to apply only for $r = R_1$ and $r = R_a$, i.e. for $\bar{h} = \bar{H}$ and $\bar{h} = 0$. For $r = R_a$, i.e. for $\bar{h} = 0$, however, the power series in question are reduced to their absolute term; thus we arrive at the following relationships between the coefficients of these developments, obtained still without the approximation method:

$$MU_{no} - U_{n1} = 0 \qquad [4.6a]$$

$$M\hat{U}_{no} - \hat{U}_{n1} + M\bar{\beta}_n\hat{W}_{no} = 0 \qquad [4.7a]$$

$$\bar{\beta}_n\hat{U}_{no} - \hat{W}_{n1} = 0 \qquad [4.8a]$$

$$M\hat{A}_{ko} - \hat{A}_{k1} + M\bar{\gamma}_k\hat{C}_{ko} = 0 \qquad [4.7b]$$

$$\bar{\gamma}_k\hat{A}_{ko} + \hat{C}_{k1} = 0 \qquad [4.8b]$$

These relationships imply that in the power series into which the left sides of the equations [3.10a], [3.11a]...can be developed, the absolute term disappears every time. If we take this into consideration and if, in order to avoid higher values for the subscript ν than $\nu = 2$, we break off the series after the term with $\bar{h}^1$ (the thinner the tube is, the more justified we are in doing this), then we obtain from the above equations with $r = R_1$, i.e. with $\bar{h} = \bar{H}$ the following approximation formulae:

$$\bar{H} \cdot (MU_{no} + MU_{n1} - 2U_{n2}) \approx -R_a \cdot P_n/\theta \qquad [4.9a]$$

$$M\hat{U}_{no} + M\hat{U}_{n1} - 2\hat{U}_{n2} + M\bar{\beta}_n\hat{W}_{n1} \approx 0 \qquad [4.10a]$$

$$\bar{\beta}_n\hat{U}_{n1} - 2\hat{W}_{n2} \approx 0 \qquad [4.11a]$$

$$M\hat{A}_{ko} + M\hat{A}_{k1} - 2\hat{A}_{k2} + M\bar{\gamma}_k\hat{C}_{k1} \approx 0 \qquad [4.10b]$$

$$\bar{\gamma}_k\hat{A}_{k1} + 2\hat{C}_{k2} \approx 0 \qquad [4.11b]$$

For the three coefficients U_{no}, U_{n1}, and U_{n2} of the power-series development of $U_n(r)$ the three linear equations [4.2a], [4.6a], and [4.9a] form a non-homogeneous system from which these coefficients can be fully determined. From the (altogether) six homogeneous linear equations [4.3a], [4.7a], [4.8a], [4.10a], [4.11a] the relationship of the six coefficients $\hat{U}_{no}$ to $\hat{W}_{n2}$ can be determined; as can also the relationship of the six coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$ from the (altogether) six homogeneous linear equations [4.3b], [4.7b], [4.8b], [4.10b], [4.11b].

The constant $\bar{\beta}_n$ results (if the parameter α_n is given) from the disappearance of the determinant of the above (homogeneous) system of equations for the six coefficients $\hat{U}_{no}$ to $\hat{W}_{n2}$. A simple calculation yields the relationship

$$\bar{\beta}_n = \frac{\bar{\alpha}_n}{\sqrt{1 - M}} \cdot \sqrt{\frac{\bar{\alpha}_n^2 + 1 - M^2}{\bar{\alpha}_n^2 \cdot (1 + M) + (1 - M)(1 + 2M)}} \qquad [4.12a]$$

(Since $\bar{\alpha}_n \geq 0$ and $0 < M < 1$, $\bar{\beta}_n$ is real and positive or equal to zero.)

From the disappearance of the determinant of the above (homogeneous) system of equations for the six coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$, a relationship results between the constants $\bar{\gamma}_k$ and $\bar{\omega}_k$ which is very similar to the relationship [4.12a] between $\bar{\beta}_n$ and $\bar{\alpha}_n$. Since, in view of the boundary conditions [3.4b] applying to the tube ends, the constant γ_k was fixed by [3.13'b], however, which for $\bar{\gamma}_k$ ($= \gamma_k \cdot R_a$) resulted in the relation

$$\bar{\gamma}_k = \frac{\pi}{2} \cdot \frac{2k+1}{\bar{L}} \quad \text{with} \quad \bar{L} = \frac{L}{R_a} \quad \text{and} \quad k = 0,1,2,\ldots\infty \qquad [4.13]$$

there follows for $\bar{\omega}_k^2$, after a simple calculation, the quadratic equation

$$\bar{\omega}_k^4 - (1 - M^2)(1 + \bar{\gamma}_k^2)\bar{\omega}_k^2 + (1 - M)^2(1 + 2M)\bar{\gamma}_k^2 = 0$$

Thus, if the subscript k is retained, we obtain two different values for $\bar{\omega}_k^2$; and for $k = 0,1,2,\ldots\infty$ there are accordingly two series of natural frequencies ω_k. The two positive roots of the above qualifying equation for $\bar{\omega}_k$ are

$$\bar{\omega}_k^{I,II} = +\sqrt{\frac{1}{2}(1 - M^2) \cdot \left[1 + \bar{\gamma}_k^2 \mp \sqrt{1 - 2\bar{\gamma}_k^2 \cdot \frac{1 + 2M - M^2}{(1 + M)^2} + \bar{\gamma}_k^4}\right]} \qquad [4.12b]$$

Considering [4.13] one may derive from this for $k \gg 1$:

$$(\bar{\omega}_k^I)_{k\gg 1} \approx \sqrt{\frac{1 - M}{1 + M}(1 + 2M)} \quad , \quad (\bar{\omega}_k^{II})_{k\gg 1} \approx \frac{\pi}{2} \cdot \frac{2k+1}{\bar{L}} \sqrt{1 - M^2} \qquad [4.14]$$

With increasing subscript k the series of the $\bar{\omega}_k^I$ thus tends toward a finite terminal value, the series of the $\bar{\omega}_k^{II}$, however, tends to the value ∞. The lowest natural frequency is given by

$$\omega_o^I = \frac{1}{R_a} \cdot \sqrt{\frac{\theta}{\rho}} \cdot \sqrt{\frac{1}{2}(1 - M^2) \cdot \left[1 + \bar{\gamma}_o^2 - \sqrt{1 - 2\bar{\gamma}_o^2 \frac{1 + 2M - M^2}{(1 + M)^2} + \bar{\gamma}_o^4}\right]} \qquad [4.12'b]$$

with $\bar{\gamma}_o = \pi/2\bar{L}$; for the sake of simplicity it shall be designated only by ω_o in the following discussion--as was done in the introduction.

Of the six coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$ or of the six coefficients $\hat{U}_{no}$ to $\hat{W}_{n2}$ we have thus far only determined their relationship. The initial conditions [3.2]* are used, for the complete determination of the coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$. For the complete determination of the coefficients $\hat{U}_{no}$ to $\hat{W}_{n2}$, however, we can go back to equation [3.13a] according to considerations in Section 3; this equation is supposed to have validity in the entire

*Cf. again the annotation on pages 10 and 11 (under consideration of the fact that there are two series of natural frequencies ω_k). The relationship of the coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$ is the same as that of the coefficients $\hat{A}_{ko}$ to $\hat{C}_{k2}$.

interval $R_1 \leqq r \leqq R_a$, i.e., in the entire interval $0 \leqq \bar{h} \leqq \bar{H}$; at least, however, it must be satisfied for $r = R_a$, i.e. for $\bar{h} = 0$. After carrying out the power-series development, we obtain from [3.13a] by comparison of coefficients

$$M \cdot (U_{no} - U_{n1}) + [M \cdot (\hat{U}_{no} - \hat{U}_{n1})] + \bar{\beta}_n \hat{W}_{no}] \cdot \cosh(\bar{\beta}_n \bar{L}) = 0 \quad [4.15]$$

The equations derived thus far in this section suffice to determine the first three, or two, terms respectively of the power-series developments for the forced parts of the displacements or stresses. In the final result we retain from these developments only the absolute terms.* [In the case of $\sigma_r^{(p)}$ or $\tau_{rz}^{(p)}$ this absolute term disappears. In the case of $\sigma_r^{(p)}$ we, therefore, still indicate the term by $\bar{h}'$; in the case of $\tau_{rz}^{(p)}$, however, this term also is equal to zero.]

After a simple calculation, we can accordingly set up the following approximation formulae for the forced parts of the displacements and stresses

$$u^{(p)} = \frac{R_a}{\bar{H}\theta} \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\bar{\alpha}_n^2 + 1 - M^2} \left[1 + \frac{M^2 \cdot (1 - M)\cosh(\beta_n Z)/\cosh(\beta_n L)}{\bar{\alpha}_n^2 \cdot (1 + M) + (1 - M)(1 + 2M)}\right]$$

$$w^{(p)} = -\frac{R_a}{\bar{H}\theta} \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\sqrt{\bar{\alpha}_n^2 + 1 - M^2}} \cdot \frac{1}{\bar{\alpha}_n} \cdot \frac{M \cdot \sqrt{1 - M} \cdot \cosh(\beta_n Z)/\cosh(\beta_n L)}{\sqrt{\bar{\alpha}_n^2 \cdot (1 + M) + (1 - M)(1 + 2M)}} \quad [4.16]$$

and

$$\sigma_\phi^{(p)} = \frac{1}{\bar{H}} \cdot \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\bar{\alpha}_n^2 + 1 - M^2} \left[1 - M^2 - \bar{\alpha}_n^2 \cdot \frac{M^2 \cdot (1 - M)\cosh(\beta_n Z)/\cosh(\beta_n L)}{\bar{\alpha}_n^2 \cdot (1 + M) + (1 - M)(1 + 2M)}\right]$$

$$\sigma_r^{(p)} = -\frac{\bar{h}}{\bar{H}} \sum_{n=0}^{N} P_n e^{-\alpha_n t}$$

$$\sigma_z^{(p)} = \frac{1}{\bar{H}} \cdot M \cdot (1 - M) \cdot \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\bar{\alpha}_n^2 + 1 - M^2} \cdot \left[1 - \cosh(\beta_n Z)/\cosh(\beta_n L)\right] \quad [4.17]$$

$$\tau_{rz}^{(p)} = 0$$

In contrast to this, the free parts of the displacements and stresses are extremely complicated;** their numerical calculation, therefore,

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*Compare with this the remarks at the end of Section 5.

**Exceptions to this are only radial stress and shear stress; $\sigma_r^{(f)} \approx 0$ and $\tau_{rz}^{(f)} \approx 0$.

would be quite protracted. Nevertheless, we can already answer the fundamental questions of the problem here under discussion, if we do not know anything else of the free vibrations but their frequencies, and particularly the lowest natural frequency, thus ω_0 ($= \omega_0^I$). This will manifest itself in the following section where we shall apply the information here obtained to the tube dealt with in report II. Before we turn to that task, we still state the radial displacement $u_n^{(p)}$ or the tangential stress $\sigma_{\phi,n}^{(p)}$ for the special case $\alpha_n = 0$. Since $\bar{\beta}_n$, according to [4.12a], disappears with $\bar{\alpha}_n$, we obtain from [4.16] or [4.17]

$$\left[u_n^{(p)}\right]_{\alpha_n = 0} = \frac{R_a}{\bar{H}} \cdot \frac{P_n}{\theta} \cdot \frac{1 + M}{(1 - M)(1 + 2M)} \qquad [4.16']$$

or

$$\left[\sigma_{\phi,n}^{(p)}\right]_{\alpha_n = 0} = \frac{P_n}{\bar{H}} \qquad [4.17']$$

It follows that for a static internal pressure $p = p_{static}$ the radial displacement

$$u_{static} = \frac{R_a}{\bar{H}} \cdot \frac{p_{static}}{\theta} \cdot \frac{1 + M}{(1 - M)(1 + 2M)} \qquad [4.16'']$$

or the tangential stress

$$\sigma_{\phi\, static} = \frac{1}{\bar{H}} p_{static} \qquad [4.17'']$$

results.

5. NUMERICAL EXAMPLES

We shall limit the following numerical considerations to the forced parts of the radial displacement or of the tangential stress as it applies in the middle of the tube, i.e. at $z = \pm L$. For these quantities, we obtain from the approximation formulae [4.16] and [4.17] the expressions

$$[u^{(p)}]_{z=0} = \frac{R_a}{\bar{H}\theta} \cdot \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\bar{\alpha}_n^2 + 1 - M^2} \left[1 + \frac{M^2 \cdot (1 - M)/\cosh(\bar{\beta}_n \bar{L})}{\bar{\alpha}_n^2(1 + M) + (1 - M)(1 + 2M)}\right] \qquad [5.1a]$$

or

$$[\sigma_\phi^{(p)}]_{z=0} = \frac{1}{\bar{H}} \cdot \sum_{n=0}^{N} \frac{P_n e^{-\alpha_n t}}{\bar{\alpha}_n^2 + 1 - M^2} \left[1 - M^2 - \bar{\alpha}_n^2 \cdot \frac{M^2 \cdot (1 - M)/\cosh(\bar{\beta}_n \bar{L})}{\bar{\alpha}_n^2(1 + M) + (1 - M)(1 + 2M)}\right]$$

[5.1b]

with

$$\bar{\alpha}_n = \alpha_n \cdot R_a \sqrt{\frac{\rho}{\theta}} \qquad [5.1']$$

and

$$\bar{\beta}_n = \frac{\bar{\alpha}_n}{\sqrt{1 - M}} \cdot \sqrt{\frac{\bar{\alpha}_n^2 + 1 - M^2}{\bar{\alpha}_n^2 \cdot (1 + M) + (1 - M)(1 + 2M)}} \qquad [5.1'']$$

(cf. [4.12a]).

We presuppose for the internal pressure p(t) the form

$$p(t) = P \cdot (e^{-\alpha t} - e^{-2\alpha t}) \text{ with } \alpha > 0 \qquad [5.2]$$

(cf. [1.1] and Figure 1) on which the numerical considerations were likewise based in reports I and II. Equation [5.2] follows from the more general form [3.5] inasmuch as in the latter the following equalities are assumed:

$$N = 1, \; P_0 = P, \; P_1 = -P, \; \alpha_0 = \alpha, \; \alpha_1 = 2\alpha$$

For a specific time value t, $[u^{(p)}]_{z=0}$ or $[\sigma_\phi^{(p)}]_{z=0}$ is to be compared in each case with that value u_{static} or $\sigma_{\phi\,static}$ which would result under static internal pressure in the amount of $p_{static} = [p(t)]_{max}$, i.e. according to [5.2], $p_{static} = P/4$ (cf. [1.7c]). From [4.16"] or [4.17"] there follow for these quantities of comparison the expressions

$$u_{static} = \frac{R_a}{\bar{H}} \cdot \frac{P}{4\theta} \cdot \frac{1 + M}{(1 - M)(1 + 2M)} \qquad [5.3a]$$

or

$$\sigma_{\phi\,static} = \frac{P}{4\bar{H}} \qquad [5.3b]$$

As a quantity of comparison for the velocity $[\frac{\partial u^{(p)}}{\partial t}]_{z=0}$ we shall use the expression

$$\left(\frac{R_a}{\bar{H}} \cdot \frac{P}{\theta}\right) \cdot \left(\frac{1}{R_a} \cdot \sqrt{\frac{\theta}{\rho}}\right) = \frac{P}{\bar{H}\sqrt{\theta\rho}} \qquad [5.3'a]$$

For M = 0.385 (m = 3.60) in the case of a tube, half the length of which is 4.7 times as great as the outer radius (thus at $\bar{L} = 4.7$), there results for the non-dimensional quantity $\bar{\omega}_0 = \omega_0 \cdot R_a \cdot \sqrt{\frac{\rho}{\theta}}$ from [4.12'b] the value $\bar{\omega}_0 = 0.3$. (The numerical values used here for M and $\bar{L}$ were also used in report II.)

The three parameter values $\alpha = \alpha_I$ or α_{II} or α_{III}, which were used in report II for determining the linear relationship between $\sigma_{\phi\,max}$ and α, yielded for the corresponding non-dimensional quantities $\bar{\alpha} = \alpha \cdot R_a \cdot \sqrt{\frac{\rho}{\theta}}$ with $R_a\sqrt{\frac{\rho}{\theta}} = 1.11 \cdot 10^{-5}$ sec (cf. [1.4]) the values

$$\bar{\alpha}_{I} = 0.35 \cdot 10^{-2},\ \bar{\alpha}_{II} = 0.35 \cdot 10^{-1},\ \bar{\alpha}_{III} = 0.35$$

Hence, the parameter $\alpha_{III}(= \bar{\alpha}_{III} \cdot \frac{1}{R_a} \cdot \sqrt{\frac{\theta}{\rho}})$ is already of the order of magnitude of the lowest natural frequency $\omega_0 (= \bar{\omega}_0 \cdot \frac{1}{R_a} \cdot \sqrt{\frac{\theta}{\rho}})$ of the tube.

We have now calculated for the above values of $\bar{\alpha}$ as well as for the additional ones

$$\bar{\alpha} = 0.50,\ \bar{\alpha} = 0.70,\ \text{and}\ \bar{\alpha} = 1$$

the following quotients:

$$\frac{[\sigma_{\phi\,max}^{(p)}]_{z=0}}{\sigma_{\phi\,static}},\quad \frac{[u_{max}^{(p)}]_{z=0}}{u_{static}},$$

$$\frac{[u^{(p)}]_{z=0}^{t=0}}{u_{static}},\quad \frac{[\frac{\partial u^{(p)}}{\partial t}]_{z=0}^{t=0}}{\frac{P}{H\sqrt{\theta\rho}}}$$

(As the formulae compiled at the beginning of this section demonstrate, numerical data other than those for the quantities M, $\bar{L}$, and $\bar{\alpha}$ are not necessary for the calculation of these quotients.)

The results of this computation are tabulated below:*

$\bar{\alpha}$	$\frac{[\sigma_{\phi\,max}^{(p)}]_{z=0}}{\sigma_{\phi\,static}}$	$\frac{[u_{max}^{(p)}]_{z=0}}{u_{static}}$	$\frac{[u^{(p)}]_{z=0}^{t=0}}{u_{static}}$	$\frac{[\frac{\partial u^{(p)}}{\partial t}]_{z=0}^{t=0}}{P/(H\sqrt{\theta\rho})}$
$0.35 \cdot 10^{-2}$	1.000	1.000	0	$+4.44 \cdot 10^{-3}$
$0.35 \cdot 10^{-1}$	1.003	1.008	0.035	$+4.36 \cdot 10^{-2}$
0.35	1.20	1.16	0.95	+0.156
0.50	1.29	1.22	1.18	+0.082
0.70	1.33	1.235	1.235	-0.026
1.00	1.14	1.05	1.05	-0.128

They demonstrate clearly that the maximum value of the total tangential stress σ_ϕ can no longer increase linearly with α when $\bar{\alpha}$ has noticeably exceeded the value $\bar{\omega}_0$ (=0.3). As the second column of this table will show, the maximum value of the forced part of σ_ϕ is already smaller at $\bar{\alpha} = 1$ than at $\bar{\alpha} = 0.70$. Columns 4 and 5 show that similar deductions probably apply for the maximum value of the free part of σ_ϕ also (which cannot be calculated directly). Since the free vibration derives its energy from the initial values of $u^{(p)}$ and $\partial u^{(p)}/\partial t$, a continuous increase of the initial values of $u^{(p)}$ and $\partial u^{(p)}/\partial t$ also would have to result if a continuous increase of

*For α_I, α_{II}, and α_{III} the values for the two quotients mentioned first coincide well with those which are on the graphs attached to report II.

$\sigma_{\phi\,max}^{(f)}$ with α should take place. The initial value of $u^{(p)}$, however, is already smaller at $\bar{\alpha} = 1$ than at $\bar{\alpha} = 0.70$, and the initial value of $\partial u^{(p)}/\partial t$ even goes through zero between $\bar{\alpha} = 0.50$ and $\bar{\alpha} = 0.70$.

Thus the statements we made in Section 1 appear to be proved. Now we shall suggest briefly the reasons why we have carried out our calculations only up to the value $\bar{\alpha} = 1$ and not at least up to the value $\bar{\alpha}_{IV} = 3.5$ (thus $\alpha_{IV} > 10\omega_0$) which appears in the table of report II. For this purpose we go back to the equations [4.2a], [4.6a], and [4.9a] from which the coefficients U_{no}, U_{n1}, and U_{n2} of the power-series development of $U_n(r)$ can be determined. From these we obtain

$$U_{n1} = MU_{no} \quad \text{and} \quad 2U_{n2} = (\bar{\alpha}_n^2 + 1 + M)U_{no}$$

with

$$U_{no} = \frac{R_a}{\bar{H}} \cdot \frac{P_n}{\theta} \cdot \frac{1}{\bar{\alpha}_n^2 + 1 - M^2}$$

The approximation formulae [5.1a] or [5.1b] for the forced radial displacement or tangential stress resulted from [3.6a] and [3.7a] after carrying out the power-series developments by ignoring in these power series even in the final result already those terms which contain coefficients with the subscript $\nu = 2$, thus especially the above coefficient U_{n2}. This is no longer permissible as soon as the parameter $\alpha_n \gg 1$. In this case, as a matter of fact, $U_{n2} \gg U_{no}$, and the terms which contain this (large) coefficient U_{n2} can, despite the smallness of $\bar{h}$ (or $\bar{H}$) no longer be neglected then as compared with terms with a lower power of $\bar{h}$.*

6. SUMMARY

The problem concerning the precise relationship between the increase of the internal pressure p(t) and the maximum value of the tangential stress σ_ϕ in a (thin-walled) tube, which was raised by the reports I and II mentioned in Section 1, is being brought up again. In this investigation we are particularly interested in the question whether a linear increase of this maximum value along with the parameter α which determines the pressure increase (such as the linear increase deduced numerically for a definite tube from three examples in report II) still applies even when the parameter α has exceeded the order of magnitude of the lowest natural frequency ω_0 of the investigated tube. The formulae required for this investigation are again

*Also the formulae derived in Section 2 for the ring do not apply to any value of α, however large.

set up in Sections 3 and 4. In applying the discovered results it is being proved in an indirect manner in Section 5 that the maximum value of the tangential stress can no longer increase along with the parameter α as soon as α has exceeded the order of magnitude of ω_0. At the same time, it is also shown that a calculation of σ_ϕ, according to the approximation formulae derived here or previously in report I, is no longer permissible when $\alpha \gg \omega_0$, i.e. when the pressure increases so sharply that it has a "shock-like" effect upon the tube.

As we have shown in Section 2, in the case of a ring (as a special case of the thin-walled tube) the fundamental questions of our problem can be dealt with much more clearly than in the case of a tube of finite length. Hence, in the case of the ring we are able to give an upper limit for the quotient $\sigma_{\phi\,max}/\sigma_{\phi\,static}$ where $\sigma_{\phi\,static}$ represents that value of σ_ϕ which would result under static internal pressure in the amount of $p_{static} = [p(t)]_{max}$. This upper limit reaches its maximum value, 8/3, at $\alpha = \omega/\sqrt{2}$.