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THE ELASTICITY PROBLEM FOR THIN SHELLS OF TOROIDAL, SPHERICAL, OR CONICAL SHAPE

by

Ernst Meissner

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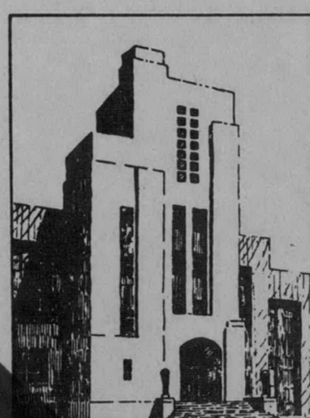

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THE ELASTICITY PROBLEM FOR THIN SHELLS OF TOROIDAL, SPHERICAL, OR CONICAL SHAPE

(Das Elastizitätsproblem für dünne Schalen von Ringflächen-, Kugel- oder Kegelform)

by

Ernst Meissner

(Physik. Zeitschr. XIV, 1913)

Translated by William A. Nash, Ph.D.

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THE ELASTICITY PROBLEM FOR THIN SHELLS OF TOROIDAL, SPHERICAL, OR CONICAL SHAPE

The elastic theory of thin shells which are formed as surfaces of revolution and which are loaded with axial symmetry, requires the integration of a total differential equation of the fourth order. Even in the simple cases this equation is difficult to solve.

In the present paper I intend to show that a reduction to a second-order differential equation occurs if the curvature of the meridional surface is constant, such as in the case of the sphere, cone, and torus. Also, I will develop a method for the integration of this equation and thereby make possible the strength calculation of such shells. This is of importance in industry, since pressure vessels of these shapes are frequently utilized. Also, the theory of spherical arches is consequently solved.

1

Let P (α) be a point on the meridian of the middle surface of the shell, α the angle that the normal to the meridian makes with the axis of rotation, R_1 the radius of curvature of the meridian, and PC = R_2 the second principal radius of curvature of the surface at P.* We pass through P an x-axis along the tangent to the meridian, a y-axis along the tangent to the parallel circle, and a z-axis along the normal to the inner surface. The shell has the constant thickness 2h, and the loading per unit area of the middle surface has the components X, Y = 0, Z, which may be given functions of α. If we consider a section normal to the middle surface of the shell along the parallel circle through P, then the following forces act on an element of unit length of this shell:

(a) Tensile stresses uniformly distributed over the shell thickness and having the resultant T_1,

(b) Normal (bending) stresses whose magnitudes increase in proportion to the distance from the middle surface of the shell; these stresses are statically equivalent to a pair of forces of moment G_1,

(c) Shearing stresses in the z-direction with the resultant N (see Figure).

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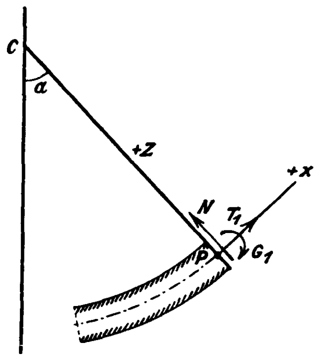
*I employ essentially the notations and formulas of Love-Timpe, "Lehrbuch der Elastizität," Chapter XXIV, Leipzig, Teubner, 1907. In the notes, the figures given in brackets [] and preceded by L-T refer to the formulas of the theory developed there.



On a meridional section through P there act analogous stresses of resultant T_2 and the moment G_2 per unit length. Here the symmetry is lacking because of the shear.

Between the five stress components T_1, G_1, N, T_2, G_2 there exist the three equilibrium equations:*

$$\left.\begin{aligned}(T_1 R_2 \sin\alpha)' - T_2 R_1 \cos\alpha - N R_2 \sin\alpha + R_1 R_2 \sin\alpha \cdot X = 0,\\ (N R_2 \sin\alpha)' + T_1 R_2 \sin\alpha + T_2 R_1 \sin\alpha + R_1 R_2 \sin\alpha\, Z = 0,\\ (G_1 R_2 \sin\alpha)' - G_2 R_1 \cos\alpha - N R_1 R_2 \sin\alpha = 0.\end{aligned}\right\}\quad [1]$$

The prime here as well as in the following denotes differentiation with respect to α.

From the first two equations one obtains the integral

$$R_2 \sin\alpha \cdot (T_1 \sin\alpha + N \cos\alpha) = -F(\alpha),$$

where

$$F(\alpha) = \int R_1 R_2 \sin\alpha [X \sin\alpha + Z \cos\alpha] d\alpha + \text{constant} \qquad [2]$$

The magnitude of the constant results from the static interpretation of the integral.

Let u, w be the displacements that the point P (α) undergoes in the x and z directions respectively, and let ϵ_1, ϵ_2 be the normal strains at P in the x and y directions respectively. Then**

$$\varepsilon_1 = \frac{u' - w}{R_1}, \qquad \varepsilon_2 = \frac{u \cot\alpha - w}{R_2}, \qquad [3]$$

and the elastic relations yield†

$$\varepsilon_1 = \frac{1}{2Eh}(T_1 - \sigma T_2), \quad \varepsilon_2 = \frac{1}{2Eh}(T_2 - \sigma T_1). \qquad [4]$$

Here E is the modulus of elasticity of the material and σ denotes Poisson's ratio.

In addition we may put††

$$\varkappa_1 = \frac{1}{R_1}\left(\frac{u + w'}{R_1}\right)', \quad \varkappa_2 = \frac{\cot\alpha}{R_2}\left(\frac{u + w'}{R_1}\right). \qquad [5]$$

---

*L-T [45], [46]
**L-T [21]
†L-T [36']
††L-T [26]

The quantities $\varkappa_1, \varkappa_2$, the so-called changes of curvature, are related to the moments of the stresses as follows:*

$$\left.\begin{aligned} G_1 = -D[\varkappa_1 + \sigma\varkappa_2],\ G_2 = -D(\varkappa_2 + \sigma\varkappa_1), \\ D = \frac{2Eh^3}{3(1-\sigma)}.** \end{aligned}\right\} \qquad [6]$$

By means of Equations [3], [4], [5], [6] the displacements u, w can be introduced in the equilibrium Equations [1]. If one then eliminates N, there will result two simultaneous differential equations for u and w as functions of α.

2

For a spherical shell Reissner[1] has obtained a symmetry in the fundamental equation by introducing stresses and deformations in place of the displacements u, w. This can also be obtained for the case of the general surface of revolution and, as is shown here, is of fundamental importance.

As a fundamental variable we choose

$$V = NR_2. \qquad [7]$$

The first two equilibrium Equations [1], then become

$$\left.\begin{aligned} T_1 &= -\frac{1}{R_2}\cot\alpha\cdot V - \frac{1}{R_2}\frac{F(\alpha)}{\sin^2\alpha}, \\ T_2 &= -\frac{1}{R_1}\cdot V' + H(\alpha), \end{aligned}\right\} \qquad [8]$$

where

$$H = \frac{F(\alpha)}{R_1\sin^2\alpha} - R_2 Z \qquad [8']$$

H depends upon the loading.

As a second variable we choose the expression

$$U^* = R_2\varkappa_2\tan\alpha = \frac{u + w'}{R_1}, \qquad [9]$$

*L-T [37']

**Translator's Note: This equation evidently contains a misprint; it should read

$$D = \frac{2E\,h^3}{3(1-\sigma^2)}$$

[1]References are listed on page 15.

so that [5] becomes

$$\varkappa_1 = \frac{1}{R_1} U^{*\prime}, \quad \varkappa_2 = \frac{\cot \alpha}{R_2} U^*. \qquad [10]$$

The last equilibrium Equation, [1], then becomes

$$0 = \frac{R_2}{R_1} U^{*\prime\prime} + \left\{ \left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \cot \alpha \right\} U^{*\prime} - U \left\{ \frac{R_1}{R_2} \cot^2 \alpha + \sigma \right\} - \frac{V R_1}{D}.$$

Now we introduce the linear homogeneous differential operator

$$L(\;) = \frac{1}{\sin \alpha} \frac{d}{d\alpha} \left[\frac{R_2 \sin \alpha}{R_1} \frac{d(\;)}{d\alpha} \right] - \frac{R_1}{R_2} \cot^2 \alpha \cdot (\;) \qquad [11]$$

The preceding equation may now be written more simply in the form

$$L(U^*) - \sigma U^* = -\frac{R_1}{D} \cdot V. \qquad [I]$$

A second relation between U* and V gives the condition of compatibility, which is found as follows:

From [3] we have

$$u = \sin \alpha \cdot \int \frac{\varepsilon_1 R_1 - \varepsilon_2 R_2}{\sin \alpha} d\alpha,$$

$$w = \cos \alpha \cdot \int \frac{\varepsilon_1 R_1 - \varepsilon_2 R_2}{\sin \alpha} d\alpha - R_2 \varepsilon_2$$

and consequently from [9] we find that

$$U^* = \frac{1}{R_1} [-(R_2 \varepsilon_2)' + \cot \alpha (R_1 \varepsilon_1 - R_2 \varepsilon_2)],$$

which because of [4] transforms into

$$2Eh \cdot U^* \cdot R_1 = -(R_2 T_2)' + \sigma (R_2 T_1)' + \cot \alpha \{R_1 T_1 - \sigma R_1 T_2 - R_2 T_2 + \sigma R_2 T_1\}.$$

If, by means of Equation [8], one now introduces the quantity V, then

$$2EhR_1 \cdot U^* = \left(\frac{R_2}{R_1} \right) V'' + \left[\left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \cot \alpha \right] V' - V \left[\frac{R_1}{R_2} \cot^2 \alpha - \sigma \right] + \Phi(\alpha),$$

where

$$\Phi = \frac{F(\alpha)}{\sin^2 \alpha} \left\{ \cot \alpha \left(\frac{R_2}{R_1} - \frac{R_1}{R_2} \right) - \left(\frac{R_2}{R_1} \right)' \right\} + (R_2^2 Z)' - R_2 (R_2 + \sigma R_1) X \qquad [12]$$

Actually, the differential expression analogous to that of Equation [I] appears here, so that this relation may be written in the form

$$2Eh \cdot R_1 \cdot U^* = L(V) + \sigma V + \Phi. \qquad [II]$$

For normalization, we write

$$U^* = i \cdot \frac{U}{\lambda \cdot D}, \quad \lambda = \frac{\sqrt{3(1-\sigma^2)}}{h}, \quad i = \sqrt{-1}. \qquad [13]$$

Then Equations [I] and [II] become

$$L(U) - \sigma U = i\lambda R_1 \cdot V, \qquad [I']$$

$$L(V) + \sigma V = i\lambda R_1 \cdot U - \Phi. \qquad [II']$$

The quantities U and V are determined from these two simultaneous differential equations.

3

Elementary particular solutions of [I'] and [II'] can be given for the sphere and cone for practically all important cases of loading. In the following the case of the homogeneous system of equations ($\Phi = 0$) will be treated first of all. This corresponds to the unloaded shell.

For the case $\Phi = 0$, the elimination of U gives for V the fourth-order homogeneous differential equation

$$L\left(\frac{L(V)}{R_1}\right) - \sigma L\left(\frac{V}{R_1}\right) + \sigma \frac{L(V)}{R_1} + (\lambda^2 R_1^2 - \sigma^2)\frac{V}{R_1} = 0 \qquad [III]$$

An analogous equation holds for U.

If we now assume that the meridian is a circle (which is true of the sphere, the torus, and the cone in the limiting case), then R_1 is constant, and if in addition

$$\lambda^2 R_1^2 - \sigma^2 = \varkappa^2$$

then [III] now becomes simply

$$L L(V) + \varkappa^2 V = 0, \qquad [IV]$$

The same equation is also satisfied by U.

However, this equation separates into the conjugate second-order equations

$$L(V) + i\varkappa V = 0, \qquad [V_1]$$

$$L(V) - i\varkappa V = 0, \qquad [V_2]$$

The entire elastic problem depends upon the integration of these equations. The solutions of $[V_2]$ are simply the complex conjugates of the solutions of $[V_1]$ so that one can essentially restrict oneself to $[V_1]$.

For the torus, the center of whose meridian circle is at the distance $a = \mu R_1$ from the axis of rotation,

$$R_2 = \frac{a + R_1 \sin\alpha}{\sin\alpha} = R_1 \frac{\sin\alpha + \mu}{\sin\alpha},$$

$$L(V) = \frac{\mu + \sin\alpha}{\sin\alpha} V'' + \cot\alpha \cdot V' - \frac{\cos^2\alpha}{\sin\alpha(\sin\alpha + \mu)} \cdot V,$$

and if one substitutes $x = \sin\alpha$, then $[V_1]$ becomes

$$(\mu + x)^2 (1 - x^2)^2 \frac{d^2V}{dx^2} + (\mu + x)(1 - x^2)[1 - x^2 - x(\mu + x)] \frac{dV}{dx} - (1 - x^2)[1 - x^2 - i\varkappa x(\mu + x)] V = 0.$$

This is a linear differential equation of the Fuchsian type with singularities at $x = \pm 1, -\mu$, and ∞.

The exponents corresponding to $x = \pm 1$ are equal to 0, 1/2; those corresponding to $x = -\mu$ are equal to ± 1, and those corresponding to $x = \infty$ are the roots of

$$\beta^2 - \beta - (1 + i\varkappa) = 0.$$

If the meridional circle does not intersect the axis, then the singular point $x = -\mu$ lies outside of the points ± 1, and one can give the solution as a power series without difficulty. This series converges for $-1 < x < 1$. Also, by application of the general theory, the nature of the function V can be investigated.

For brevity, I prefer to discuss completely only the simpler cases of the spherical and conical shells and to withhold the discussion of the torus until later.

4

The Spherical Shell

For the sphere:

$$R_1 = R_2 = R; \ \mu = 0$$

$$L(V) = V'' + \cot\alpha \cdot V' - \cot^2\alpha \cdot V,$$

and from [12] we have

$$\Phi = R^2[Z' - (1 + \sigma)X], \qquad [14]$$

so that the nonhomogeneous equations [I'] and [II'] become

$$L(U) - \sigma U = i\lambda R V, \qquad [15_1]$$

$$L(V) + \sigma V = i\lambda R U - R^2[Z' - (1 + \sigma)X]. \qquad [15_2]$$

We must find particular solutions of these equations for technically important types of loading, and then, in order to be able to satisfy the general boundary conditions, we will superpose the solution for the unloaded shell. The latter necessitates carrying out the integration of [V_1] which becomes

$$V'' + V' \cot\alpha - V\cot^2\alpha + i\varkappa V = 0. \qquad [16]$$

5

Particular Solutions

(a) Constant surface pressure p: In this case $X = 0$, $Z = p$, $\Phi = 0$, and U = o, V = o satisfy Equations [15]. This leads essentially to a closed hollow sphere under external pressure. On every section there occur uniformly distributed tensile* stresses

$$\left(T_1 = T_2 = p\frac{R}{2};\ G_1 = G_2 = N = 0\right).$$

(b) Loading due to its own weight: If γ is the specific weight and Γ is the weight of the shell per unit area of the middle surface ($\Gamma = 2h\gamma$), then we have for an axis placed vertically**

$$X = \Gamma\sin\alpha, \quad Z = \Gamma\cos\alpha,$$
$$\Phi = -R^2\Gamma(2+\sigma)\cdot\sin\alpha.$$

If we observe that

$$L(\sin\alpha) = -\sin\alpha,$$

then we easily recognize that the equations can be satisfied by means of the relation

$$U = A\sin\alpha, \quad V = B\sin\alpha$$

if A and B are appropriately chosen.

(c) Spherical shell rotating uniformly about its axis: If ω is the angular velocity, then the inertia forces are regarded as the loading, and they have the components per unit area of the surface

$$X = \tau\sin\alpha\cos\alpha, \quad Z = -\tau\sin^2\alpha,$$

where

$$\tau = 2\frac{\gamma}{g}h\omega^2 R$$

and

$$\Phi = -R^2\tau(3+\sigma)\cdot\sin\alpha\cdot\cos\alpha.$$

---

*Translator's Note: This is evidently a misprint and should read "uniformly distributed compressive stresses."

**Essentially the same equation holds if the spherical shell is accelerated (in translation) in the direction of the axis.

The observation that

$$L(\sin\alpha\cos\alpha) = -5\sin\alpha\cos\alpha,$$

shows that the expression

$$U = -RC_1\sin\alpha\cos\alpha,\quad V = -RC_2\sin\alpha\cos\alpha$$

gives a solution of [15], as soon as one puts

$$C_1 = \frac{(3+\sigma)\frac{\gamma}{g}R\omega^2}{E\left[1+\frac{5^2-\sigma^2}{3(1-\sigma^2)}\frac{h^2}{R^2}\right]},$$

$$C_2 = \frac{(3+\sigma)(5+\sigma)2h^3\frac{\gamma}{g}\cdot\omega^2}{3(1-\sigma^2)\left[1+\frac{5^2-\sigma_2}{3(1-\sigma^2)}\frac{h^2}{R^2}\right]}.$$

We then obtain from [8] and [9]

$$T_1 = C_2\cos^2\alpha,$$
$$T_2 = C_2\cos 2\alpha + \tau R\sin^2\alpha,$$
$$G_1 = DC_1[\cos 2\alpha + \sigma\cos^2\alpha],$$
$$G_2 = DC_2[\cos^2\alpha + \sigma\cos 2\alpha],$$

and recognize that this solution corresponds to a completely closed sphere free to rotate uniformly about a diameter.**

6

The Unloaded Spherical Shell

This necessitates the integration of [16]. We substitute in that equation

$$V = \sin\alpha\cdot S,\quad x = \sin^2\alpha \qquad [17]$$

and obtain

$$x(x-1)\frac{d^2S}{dx^2} + \left(\frac{5}{2}x-2\right)\frac{dS}{dx} + S\frac{1-ix}{4} = 0. \qquad [18]$$

---

*Translator's Note: The denominator of this equation evidently contains a misprint and should read

$$3(1-\sigma^2)\left[1+\frac{5^2-\sigma^2}{3(1-\sigma^2)}\frac{h^2}{R^2}\right]$$

**If the quantity $\frac{5^2-\sigma^2}{3(1-\sigma^2)}\frac{h^2}{R^2}$ is small compared with unity, the flattening of the sphere is given by $\frac{\gamma}{g}\cdot 3(2+\sigma)\cdot\frac{V_a^2}{g}$, where V_a is the peripheral velocity at the equator.

This is a hypergeometric differential equation for $S(x)$ with the exponents[2]

$$\alpha' = 0, \quad \alpha = -1 \quad \text{at } x = 0,$$
$$\gamma' = \frac{1}{2}, \quad \gamma = 0 \quad \text{at } x = 1,$$
$$\beta' = \frac{3+W}{4}, \quad \beta = \frac{3-W}{4},$$

where

$$W = \sqrt{5 + 4ix} \qquad \text{at } x = \infty .$$

A first integral is thus the hypergeometric series

$$S_1 = F\left(\frac{3+W}{4}, \frac{3-W}{4}, 2, x\right) = 1 + \frac{3^2 - W^2}{1!\,2!}x + \frac{(3^2-W^2)(7^2-W^2)}{2!\,3!}x^2 + \cdots \qquad [19]$$

A second integral is of the form

$$S_2 = \lg x \cdot S_1 + \frac{1}{x}\mathfrak{P}(x), \qquad [20]$$

where the power series $\mathfrak{P}(x)$ as well as the series [19] converges for all $|x| < 1$. The general coefficients may be given[3] in slightly simpler form than for the series [19]. By virtue of [17] the integrals [19] and [20] correspond to the two integrals V_1, V_2 of $[V_1]$. If we separate real and imaginary parts

$$V_1 = I_1 + iI_2 = \sin\alpha S_1, \; V_2 = I_3 + iI_4 = \sin\alpha S_2, \qquad [21]$$

then by virtue of $[V_1]$

$$\left.\begin{array}{ll} L(I_1) = \varkappa I_2, & L(I_3) = \varkappa I_4, \\ L(I_2) = -\varkappa I_1, & L(I_4) = -\varkappa I_3. \end{array}\right\} \qquad [22]$$

We can recognize that the general solution of [4] is of the form

$$V = c_1 I_1 + c_2 I_2 + c_3 I_3 + c_4 I_4. \qquad [23]$$

where the c_i's are constants of integration. We obtain from $[15_2]$ where $X = Z = 0$

$$i\lambda RU = 2EhRU^* = L(V) + \sigma V,$$

and then, because of [21] and [22], we find that

$$U^* = (\sigma c_1 - \varkappa c_2) I_1 + (\varkappa c_1 + \sigma c_2) I_2 + (\sigma c_3 - \varkappa c_4) I_3 + (\varkappa c_3 + \sigma c_4) I_4. \qquad [24]$$

Therefore the mathematical part of the problem is disposed of. We may obtain the values of the constants of integration c_i from the edge conditions prevailing at the boundary of the shell. If, for example, the shell is closed at the crown $(\alpha = 0)$ then the stresses there must be finite, and the integrals

I_3 and I_4 do not appear ($C_3 = C_4 = 0$). Along the free edge of the shell any boundary conditions compatible with the equilibrium conditions can be satisfied. For a shell that is open at the crown, I_3 and I_4 do appear. Therefore, one must satisfy additional boundary conditions at the edge of the hole.

From the form of the integrals I_3 and I_4 we can easily deduce the following relation:

> If the inner edge of the hole is stress-free, then the stress components T_2, G_2 assume values there such that, when the diameter of the hole decreases to zero, the values are double those values which under otherwise equal conditions would prevail at the crown of the shell without any hole.

Thus the known result for a plane plate is extended to the spherical shell. It illustrates the danger of a hole or a small crack on the strength of the crown.

7

In the previous section, the elastic theory of spherical domes is reduced to the numerical calculations appearing in the power series [19] and [20]. The region of convergence of these series is to be taken into consideration. That these series are useful in practice is shown in work by Bolle now being carried out under my supervision.* It is clear that one can make use of the relation that exists between the hypergeometric developments in powers of

$$x = \sin^2\alpha \quad \text{and} \quad 1 - x = \cos^2\alpha$$

for attainment of better convergence. Also, it is of interest to compare the results of the exact theory developed here with those of the approximate theories due to Blumenthal[4] and Reissner[1] which are developed by asymptotic integration.

8

The Conical Shell

Stodola[5] was the first to treat the problem of the conical shell, when he introduced the power-series expression for the displacement into the

*Translator's Note: This work was published as "Festigkeitsberechnung von Kugelschalen" von der Eidgenössischen Technischen Hochschule in Zürich, L. Bolle, Zürich, 1916.

differential equation. Here a simple theory for these shells results as a limiting case of the development of the second section of this paper. There, we may take $R_1\alpha = x$ and pass to the limit, $R_1 = \infty$. If β denotes the half apex angle of the cone, then

$$\lim_{R_1=\infty} \frac{1}{R_1} \cot\beta \cdot L(\ ) = x\frac{d^2(\ )}{dx^2} + \frac{d(\ )}{dx} - \frac{(\ )}{x} = \Lambda(\ ), \qquad [25]$$

$$\lim_{R_1=\infty} \frac{\Phi}{R_1} \cot\beta = -\frac{F_1(x)}{x\cos\beta} + \tan\beta \cdot \frac{d(x^2 Z)}{dx} - \sigma x X = \Phi_1, \qquad [26]$$

where

$$F_1(x) = \int x(X\cos\beta + Z\sin\beta)\,dx. \qquad [26']$$

The fundamental equations [I] and [II] become

$$\Lambda(U^*) = -V\frac{\cot\beta}{D}, \qquad [VI]$$

$$\Lambda(V) = 2Eh\cot\beta \cdot U^* - \Phi_1. \qquad [VII]$$

In addition we have, in place of [8], [9], and [10],

$$\left.\begin{aligned} T_1 &= -\frac{V}{x} - \frac{F_1}{x\cos\beta}, \\ T_2 &= -\frac{dV}{dx} - \frac{1}{\cos\beta}\frac{dF_1}{dx} + xX, \end{aligned}\right\} \qquad [27]$$

$$U^* = \frac{dw}{dx}, \quad \varkappa_1 = \frac{dU^*}{dx}, \quad \varkappa_2 = \frac{U^*}{x}, \qquad [28]$$

from which after determination of U* and V the stresses and deformations can be calculated.

9

Special Solutions

Again, for the most important cases of loading, particular solutions of [VI] and [VII] are given, whereupon we further limit ourselves to the homogeneous equations.

(a) Constant surface pressure p:

Here

$$X = 0, \quad Z = p, \quad \Phi = \frac{3}{2}\tan\beta \cdot p \cdot x + \frac{c}{x}$$

where c is a constant

We observe that for every n

$$A(x^n) = (n^2 - 1)\, x^{n-1}. \qquad [29]$$

We then immediately find the solution

$$U^* = \frac{3}{4}\,\frac{\tan^2\beta \cdot p}{Eh} \cdot x + \frac{c\,\tan\beta}{2Eh}\,\frac{1}{x}, \qquad V = 0.$$

Thus in this case no shearing force exists; at the edge of the shell eccentric normal pressure must be exerted. At the apex of the closed cone c = 0, and from [28] it follows that

$$w = p \cdot \frac{3}{\gamma}\,\frac{\tan^2\beta}{Eh} \cdot x^2,$$

The cone thus bends into a paraboloid of revolution.

(b) Cone loaded by its own weight or by axial acceleration: As in the case of the sphere, we have

$$X = \Gamma\cos\beta, \quad Z = \Gamma\sin\beta, \quad \Phi_1 = a\,x + \frac{b}{x},$$

so that again the relation

$$U = A\,x + \frac{B}{x}, \qquad V = 0$$

gives a solution.

(c) Cone rotating with uniform angular velocity about its axis: Here,

$$X = \tau\sin\beta \cdot x, \quad Z = -\tau\cos\beta \cdot x,$$
$$\left(\tau = \frac{\gamma}{g} \cdot 2h \cdot \sin\beta \cdot \omega^2\right)$$

and

$$\Phi_1 = -(3 + \sigma)\sin\beta \cdot \tau \cdot x^2 \cdot - \frac{c}{x} \cdot$$

By virtue of [14] we can determine the constants A, B, and C in

$$U^* = A\,x^2 + \frac{B}{x}, \qquad V = C\,x$$

so that the equations [VI] and [VII] are satisfied.

10

The Unloaded Conical Shell

Here we have

$$\Phi_1 = \frac{c}{x},$$

where c denotes a constant which is zero for the closed cone. By observation of [13] and putting

$$\xi = x \cdot \lambda \cot \beta,$$

$$\Delta^*(\;) = \xi \frac{d^2(\;)}{d\xi^2} + \frac{d(\;)}{d\xi} - \frac{(\;)}{\xi} = \frac{1}{\lambda \cot \beta} \cdot \Delta(\;),$$

and substituting in [VI] and [VII] the expression

$$U^* = i \cdot \frac{\lambda}{2Eh} W + \frac{c \tan\beta}{2Eh} \cdot \frac{1}{x},$$

we obtain

$$\Delta^*(W) = iV, \qquad \text{[VIII]}$$

$$\Delta^*(V) = iW. \qquad \text{[IX]}$$

In these equations only pure numbers appear as coefficients. Elimination yields

$$\Delta^* \Delta^*(V) + V = 0, \qquad [X_1]$$

$$\Delta^* \Delta^*(W) + W = 0. \qquad [X_2]$$

Equation $[X_1]$, for example, may be separated into

$$\Delta^*(V) + iV = 0, \qquad [30_1]$$

$$\Delta^*(V) - iV = 0, \qquad [30_2]$$

The problem is reduced to solving these conjugate second-order equations. They have singular points at $\xi = 0, \infty$. The exponents for $\xi = 0$ are 1. Thus there exists an integral V_1 of $[30_1]$ which can be developed in a power series convergent with respect to ξ. The use of [29] immediately leads to the equation

$$V_1 = \sum_{1}^{\infty} \frac{(-i\xi)^{\varkappa}}{(\varkappa - 1)!\,(\varkappa + 1)!} \quad (0! = 1).$$

Separating real and imaginary parts, we have

$$V_1 = I_1 + iI_2,$$

where

$$\left. \begin{aligned} I_1 &= \sum_{1}^{\infty} \frac{(-1)^{\lambda} \xi^{2\lambda}}{(2\lambda - 1)!\,(2\lambda + 1)!}, \\ I_2 &= \sum_{1}^{\infty} \frac{(-1)^{\lambda} \xi^{2\lambda - 1}}{(2\lambda - 2)!\,(2\lambda)!} \end{aligned} \right\} \qquad [31]$$

are real power series, which can be tabulated immediately.

A second integral of $[30_1]$ is of the form

$$V_2 = \log \xi \cdot V_1 + \frac{1}{\xi} \mathfrak{P}(\xi) = I_3 + iI_4 \qquad [32]$$

where $\mathfrak{P}(\xi)$ denotes a power series. Here also the functions I_3 and I_4 can be tabulated. By virtue of [30_1] the following equations hold:

$$\Lambda^*(I_1)=I_2, \quad \Lambda^*(I_2)=-I_1, \quad \Lambda^*(I_3)=I_4, \; \Lambda^*(I_4)=-I_3,$$

and the function I thus satisfies the differential equation [X_1]. Consequently

$$V=c_1 I_1 + c_2 I_2 + c_3 I_3 + c_4 I_4, \tag{33}$$

and from [IX] we have

$$W=-i\Lambda^*(V)=-i\,[c_1 I_2 - c_2 I_1 + c_3 I_4 - c_4 I_3], \tag{34}$$

whereby the problem is solved.

For a cone that is closed at the apex, $c_3 = c_4 = 0$ once again. Because of the form of the integrals I_3 and I_4 there again follows a result concerning the stresses at the hole analogous to that given for the sphere at the end of Section 6.

The elastic theory of conical shells is considerably simplified because essentially only the four pure numerical quantities I_1, ... I_4 appear, and they can be calculated once and for all.

I have examined the practicality of the method developed in regard to numerical computation while checking Stodola's work.[5] Further calculation is being undertaken at the present time by F. Dubois.

11

I close with the following remarks:

If a case of loading exists for which a particular solution cannot be stated immediately, then we must solve the nonhomogeneous system of equations [I'], [II']. If R_1 is constant, the elimination of V gives an equation of the form

$$LL(V) + \varkappa^2 V = \varphi(\alpha).$$

Here, $\phi(\alpha)$ is a known function dependent upon the load. In order to obtain from this a particular integral V_0, we seek to determine a function $\psi(\alpha)$ from the second-order differential equation

$$L(\psi) - i\varkappa\psi = \varphi$$

We can then set up the equation

$$L(V_0) + i\varkappa V_0 = \psi,$$

which is again of the second order. The problem is thus reducible in this case also.

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