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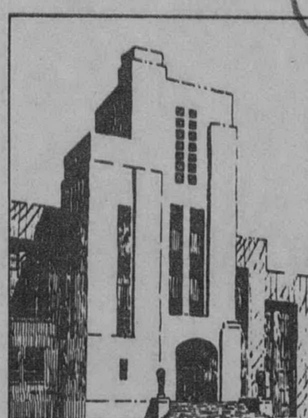
CONTRIBUTION TO THE CALCULATION OF TURBULENT BOUNDARY LAYERS

by

J. Rotta



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CONTRIBUTION TO THE CALCULATION OF TURBULENT BOUNDARY LAYERS

(BEITRAG ZUR BERECHNUNG DER TURBULENTEN GRENZSCHICHTEN)

by

J. Rotta

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CONTRIBUTION TO THE CALCULATION OF TURBULENT BOUNDARY LAYERS*

by

J. Rotta

1. INTRODUCTION

In judging the flow conditions about a body and especially in estimating its resistance to flow, the behavior of the layer of fluid adjacent to the body, which may be either laminar or turbulent, is of great importance. It is true that the physical conditions for laminar boundary layers have been clarified and the mathematical problems involved have also been solved to the point where methods of calculation are available for use in actual practice. The exact calculation of turbulent boundary layers, however, is still impossible. The well known methods of approximation[1] can be improved if one considers that the kinematic viscosity ν and the geometrical configuration of the wall (wall roughness) only influence the velocity profile near the wall in a layer δ_w which is very thin compared with the boundary-layer thickness δ and that with proper normalization the flow quantities in the remaining zone of the boundary layer appear to be almost independent of viscosity and wall roughness.

2. BOUNDARY-LAYER EQUATIONS

The following statements are limited to flows of an incompressible fluid that are steady on an average. Let the x-axis be taken parallel to the wall and let y represent the normal distance from the wall. On a two-dimensional mean flow which has in the x- and y-directions velocity components U and V averaged with respect to time, there is superposed a fluctuating turbulent motion, varying in time, with the components u, v, and w which are always three-dimensional. Integration of the boundary-layer equation for a plane wall

$$\varrho\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y} \qquad [2.1]$$

*A more detailed description is found in: J. Rotta, "Über die Theorie der turbulenten Grenzschichten," Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung, No. 1, Göttingen, 1950.

[1]References are listed on page 20.

in which p represents the time average of the static pressure, ϱ the mass density of the fluid, and

$$\tau = \varrho\left(\nu \frac{\partial U}{\partial y} - \overline{uv}\right) \qquad [2.2]$$

the shearing stress in the xy-plane, averaged with respect to time, leads, with the aid of the equation of continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \qquad [2.3]$$

and the corresponding boundary conditions

$$\begin{array}{lllll} y = 0: & U = 0; & V = 0; & \overline{u^2} = \overline{v^2} = \overline{w^2} = 0 & \\ y \to \delta: & U \to U_1; & \overline{u^2} \to 0; & \overline{v^2} \to 0, & \overline{w^2} \to 0 \end{array} \qquad [2.4]$$

to the familiar momentum equation

$$\frac{d}{dx}(U_1^2 \delta_2) + U_1 \delta_1 \frac{dU_1}{\partial x} = \frac{\tau_0}{\varrho} \qquad [2.5]$$

Here

$$\delta_1 = \int_0^\infty \left(1 - \frac{U}{U_1}\right) dy \qquad [2.6]$$

represents the displacement thickness,

$$\delta_2 = \int_0^\infty \frac{U}{U_1}\left(1 - \frac{U}{U_1}\right) dy \qquad [2.7]$$

the momentum thickness, U_1 the velocity at the outer edge of the boundary layer and τ_0 the shearing stress at the wall.

The mean value, with respect to time, of the product $\overline{uv}$ of the fluctuating orthogonal components which occur in [2.2] is to be regarded as an additional unknown which does not exist in laminar flow. We must now look for additional relations to establish a connection between this unknown $\overline{uv}$ and the remaining flow quantities. We find such a relation, for instance, in a balance of the time averaged kinetic energy of turbulence per unit mass

$$E = \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{2} \qquad [2.8]$$

where the bars again indicate the average value with respect to time. According to such a balance, first stated by L. Prandtl,[2] the convective change in the turbulent energy per unit time (the total or substantial derivative) is equal to the difference between the energy transferred from the mean flow (work of the mean stresses) and the sum of the energy transformed into heat by friction (dissipation S) and the energy exchanged with the neighboring points (diffusion of energy Q). Using the familiar boundary-layer simplifications this energy balance assumes the form

$$\underbrace{\varrho\left(U\frac{\partial E}{\partial x}+V\frac{\partial E}{\partial y}\right)}_{\substack{\text{Convective change}\\ \text{in the energy of}\\ \text{turbulence}}} = \underbrace{\tau\frac{\partial U}{\partial y}}_{\substack{\text{Energy}\\ \text{transferred}\\ \text{from the}\\ \text{mean flow}}} - \underbrace{\varrho S}_{\text{Dissipation}} - \underbrace{\varrho\frac{\partial Q}{\partial y}}_{\substack{\text{Energy}\\ \text{diffusion}}} \tag{2.9}$$

Here the dissipation per unit mass is

$$S=\nu\left[\left(\frac{\partial U}{\partial y}\right)^2+2\overline{\left(\frac{\partial u}{\partial x}\right)^2}+2\overline{\left(\frac{\partial v}{\partial y}\right)^2}+2\overline{\left(\frac{\partial w}{\partial z}\right)^2}+\overline{\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2}+\overline{\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2}+\overline{\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2}\right] \tag{2.10}$$

and the energy diffusion in the direction of the y-axis

$$Q=-\nu\left(\frac{\partial E}{\partial y}+\frac{\partial\overline{v^2}}{\partial y}\right)+\overline{v\left(\frac{u^2+v^2+w^2}{2}+\frac{\tilde{p}}{\varrho}\right)} \tag{2.11}$$

$\tilde{p}$ designates the fluctuations of pressure with respect to time. By integrating over y we can then develop an energy-integral theorem of form

$$\underbrace{\frac{1}{2}\frac{d}{dx}(U_1^3\delta_3)}_{\substack{\text{Energy decrease}\\ \text{of the mean}\\ \text{flow}}} = \underbrace{D}_{\text{Dissipation}} + \underbrace{\frac{d}{dx}\int_0^\infty UE\,dy}_{\substack{\text{Increase in the}\\ \text{turbulent energy}}} \tag{2.12}$$

where the energy thickness is

$$\delta_3=\int_0^\infty \frac{U}{U_1}\left[1-\left(\frac{U}{U_1}\right)^2\right]dy \tag{2.13}$$

and the dissipation function

$$D=\int_0^\infty S\,dy \tag{2.14}$$

While the momentum equation [2.5] has the same form for both turbulent and laminar boundary layers, the energy equation [2.12] is characteristic of the behavior of turbulent boundary layers. The energy losses of the mean flow are first essentially transformed into the kinetic energy of turbulence which in turn is then converted into heat by friction. Equation [2.12] expresses the fact that the transfer of the energy of the mean flow into the energy of turbulence and the transformation of the energy of turbulence into heat need not take place at the same point. If the fluctuation velocities are allowed to approach zero, we obtain from [2.12] the energy equation for laminar boundary layers stated by K. Wieghardt.[3]

$$\frac{1}{2}\frac{d}{dx}(U_1^3\delta_3) = \nu\int_0^\infty\left(\frac{\partial U}{\partial y}\right)^2 dy \qquad [2.12a]$$

3. SEPARATION OF THE BOUNDARY LAYER INTO A ZONE NEAR THE WALL AND AN OUTER ZONE

At distances from the wall which are small compared with the boundary-layer thickness δ, the shearing stress τ does not deviate perceptibly from the shearing stress τ_0 at the wall and the flow conditions in this zone are practically independent of the pressure gradient $\partial p/\partial x$. As is well known experimentally, the viscosity and the wall roughness exercise a direct influence on the flow processes only in a layer of thickness δ_w adjacent to the wall. If this thickness δ_w is sufficiently thin, then there are certainly distances y from the wall larger than δ_w but still very much smaller than boundary-layer thickness δ so that a universal law for the boundary-layer flow applies in this region $\delta_w \leq y \ll \delta$ according to which all flow quantities are determined by only two quantities with dimensions, viz., by the friction velocity

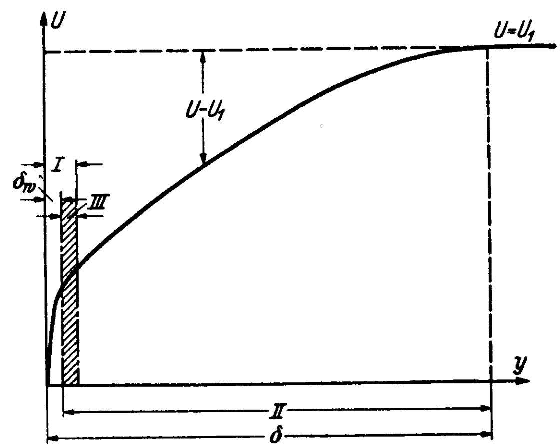
$$v^* = \sqrt{\frac{\tau_0}{\varrho}} \qquad [3.1]$$

and the absolute distance y from the reference plane which practically coincides with the wall surface. The velocity distribution of the mean flow is given here by the familiar relation[4]

$$\varkappa y\frac{\partial U}{\partial y} = v^* \qquad [3.2]$$

In this relation $\varkappa \sim 0.4$ is a universal constant.

The existence of a universal law of flow in the zone $\delta_w \leq y \ll \delta$ enables us to separate the boundary-layer flow into a part $(0 \leq y \ll \delta)$ near the wall and an outer part $(y > \delta_w)$ (see Figure 1); the former passing over asymptotically into the universal turbulent boundary-layer flow with increasing y, and the latter with decreasing y. This separation has the advantage that, after introducing the parameter v^*, the processes of flow in each zone may be investigated separately (both experimentally and theoretically) with reduction in the number of influencing factors and that both zones may be joined together again as the need arises.



	Range	Velocity Law	Dependent Upon v^* and	Not Depending Upon
I Zone near the wall	$0 \leq y \ll \delta$	$U = v^* f\left(\frac{y v^*}{\nu}\right)$	ν, wall roughness	$U_1(x)$
II Outer Zone	$\delta_w \leq y$	$U_1 - U = v^* F\left(\frac{y v^*}{\delta_1 U_1}\right)$	$U_1(x)$, (depending only on v^*/U_1)	ν, wall roughness
III Universal Law	$\delta_w \leq y \ll \delta$	$\frac{\partial U}{\partial y} = \frac{v^*}{\varkappa y}$	-	ν, wall roughness $U_1(x)$

Figure 1 - Separation of the Boundary Layer into a Zone Near the Wall and an Outer Zone

For the zone near the wall $(0 \leq y \ll \delta)$ a velocity law of the form

$$U = v^* f\left(\frac{v^* y}{\nu}\right) \qquad [3.3]$$

applies wherein the function $f(v^* y/\nu)$ generally depends on $v^* y/\nu$ and also on wall roughness. With the aid of Prandtl's mixing-length formula, existing experimental results on smooth and rough walls can be understood up to the immediate vicinity of the wall and they can be expressed by means of formulas.[5] In this connection, we are merely interested in the asymptotic form for $y > \delta_w$ which is obtained from [3.2] by integration:

$$U = v^* \left(\frac{1}{\varkappa} \ln \frac{v^* y}{\nu} + C\right) \qquad [3.4]$$

Here the quantity C is a function of wall roughness.

The outer zone ($y > \delta_w$) is independent of the viscosity and is essentially determined by two functions, viz., by the velocity distribution $U_1(x)$ at the outer edge of the boundary layer and by the friction velocity $v^*(x)$. While velocity $U_1(x)$ generally represents a given function (potential flow about a body, for example), $v^*(x)$ is a function which the boundary-layer calculation has yet to furnish. Fortunately, as will be seen later, however, the velocity profile of the outer zone with a proper normalization depends only relatively little on v^*/U_1; consequently, a separation of the turbulent boundary layer as just described may be successfully applied in a rational calculation.

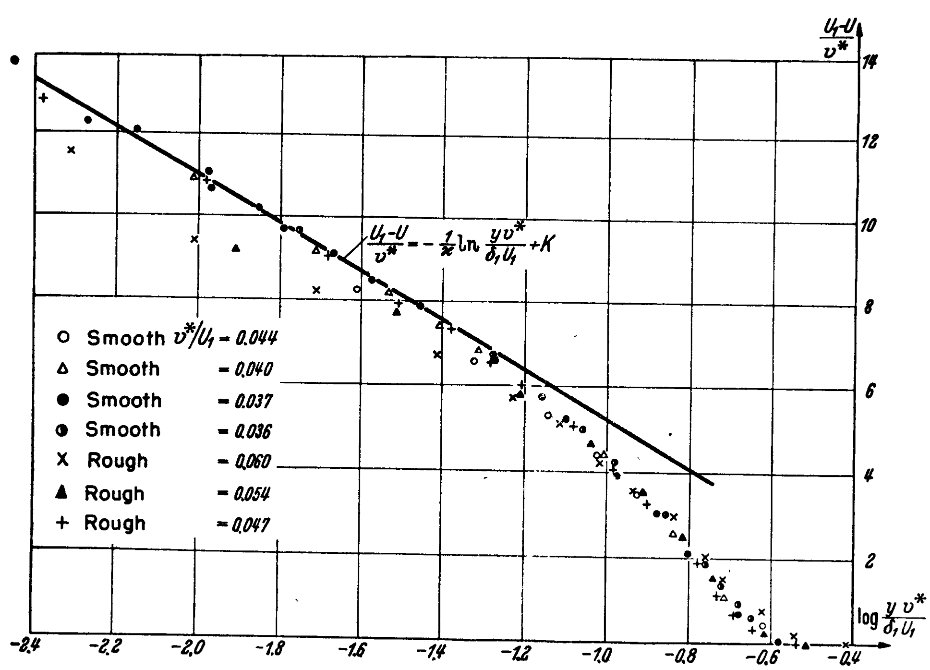


Figure 2 - Velocity Profiles at Constant Pressure, According to Measurements on Smooth Walls (by F. Schultz-Grunow) and on Rough Walls (by W. Tillmann)

The prerequisites for the separation into two zones independent of each other are satisfied in most cases. However, this is by no means a matter of course and a verification is necessary for each individual case. For this purpose, we shall indicate the following: The thickness δ_w in the case of smooth walls amounts to $\delta_w \sim 40\ \nu/v^*$; if the wall roughness is very coarse, δ_w is determined by the dimensions of the roughness elements. According to the experiments of J. Nikuradse[6] on pipes having roughnesses of sand particles, δ_w is approximately equal to the grain size of the roughness and the y-values are measured from the reference plane in which U vanishes on an average. Between the value v^*/U_1 and the local coefficient of friction c_f', there exists the following relation:

$$c_f' = 2\left(\frac{v^*}{U_1}\right)^2 \qquad [3.5]$$

4. EMPIRICAL VELOCITY PROFILES

In order to be able to calculate the boundary layer for a given velocity $U_1(x)$ according to the momentum equation [2.5] or the energy equation [2.12], we must either set up a suitable analytical formula for the velocity profile like that for laminar boundary layers or determine the necessary functional relations from test results. We shall now turn to this problem. Since a series of measurements have recently become available in which the wall shearing stresses were determined by means of a special measurement[7,8] we can process the available test material more successfully than was hitherto possible.

In order to represent the outer velocity profiles we plot the value $(U_1 - U)/v^*$ against $yv^*/\delta_1 U_1$. In doing so, the scales of the coordinates are normalized in such a manner that the integral value becomes

$$\int_0^\infty \frac{U_1 - U}{v^*}\, d\left(\frac{yv^*}{\delta_1 U_1}\right) = 1$$

as a comparison with [2.6] shows. Such a plotting of boundary-layer profiles for a flat plate without a pressure gradient[9,10] is shown in Figure 2; Figure 3 represents profiles in a rising pressure.[8]

On the basis of hydrodynamic differential equations it can be shown theoretically that in the case of turbulent flow through a pipe the velocity distribution* $(U_1 - U)/v^*$, plotted against y/r, the so-called Velocity

*U_1 in this case designates the velocity in the middle of the pipe, while r denotes the radius of the pipe.

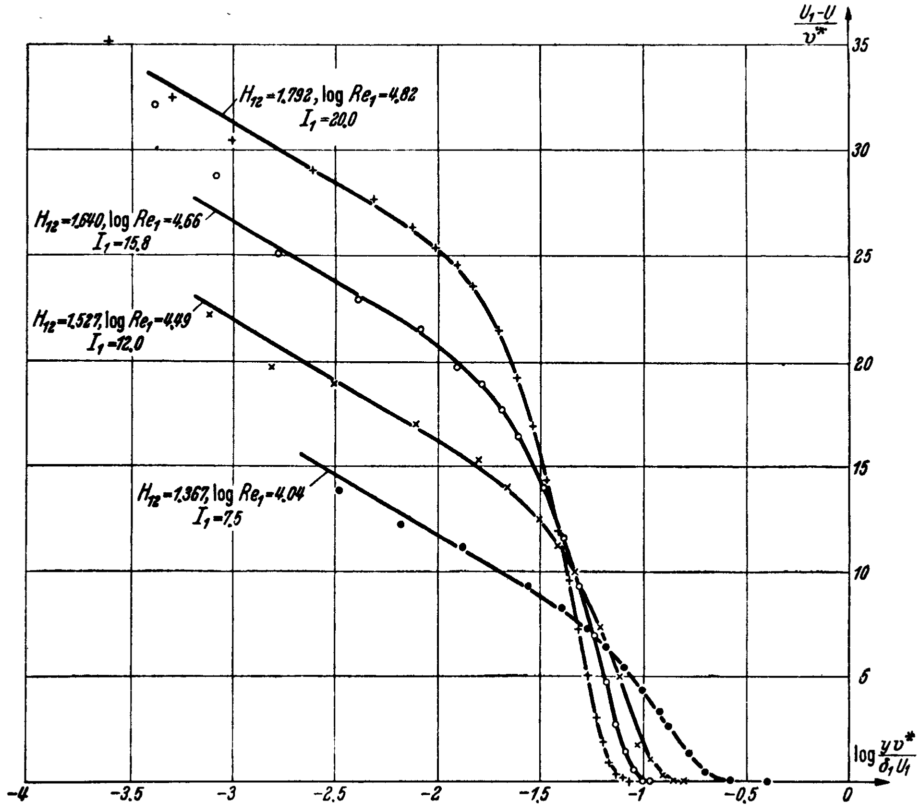


Figure 3 - Velocity Profiles in a Rising Pressure, According to Measurements by H. Ludwieg and W. Tillmann

Deficiency Law, is indeed independent of the ratio v^*/U_1. For the boundary layer on a flat plate without a pressure gradient, however, a dependence upon v^*/U_1 is to be expected even if the approach distance and the Reynolds number are sufficiently large. Upon close examination, Figure 2 actually shows such a dependence on v^*/U although it is very slight. According to Figure 3, variations in the pressure gradient, at any rate, exert a much stronger effect on the boundary-layer profiles.

Thus, the following general law applies for the outer zone of the velocity profile

$$U_1 - U = v^* F\left(\frac{yv^*}{\delta_1 U_1}\right) \qquad [4.1]$$

in which the function $F(yv^*/\delta_1 U_1)$ depends mainly on $U_1(x)$ and only in slight measure on v^*/U_1. For small distances from the wall $y \to \delta_w$, these profiles assume the asymptotic form

$$U_1 - U = v^*\left(-\frac{1}{\varkappa}\ln\frac{yv^*}{\delta_1 U_1} + K\right) \qquad [4.2]$$

where the quantity K represents a different value for each profile.

In carrying out approximate calculations it suffices to know the connection between the various characteristic quantities such as the displacement thickness δ_1, the momentum thickness δ_2, etc. The next step consists in giving these characteristic quantities a form which will permit the separate determination of the influence of viscosity and wall roughness. According to [2.7], the following formula applies to the momentum thickness δ_2

$$\delta_2 = \int_0^\infty \frac{U}{U_1}\left(1-\frac{U}{U_1}\right) dy = \int_0^\infty \left(1-\frac{U}{U_1}\right) dy - \int_0^\infty \left(1-\frac{U}{U_1}\right)^2 dy$$

which can also be written in the form

$$\delta_2 = \delta_1 \left(1 - \frac{v^*}{U_1} I_1\right) \qquad [4.3]$$

Here the value

$$I_1 = \int_0^\infty \left[\frac{U_1 - U}{v^*}\right]^2 d\left(\frac{y\,v^*}{\delta_1 U_1}\right) \qquad [4.4]$$

under the assumption made $(\delta_w \ll \delta)$, is practically independent of the velocity distribution for $y<\delta_w$. In a similar manner the energy thickness can, according to [2.13], be written in the form

$$\delta_3 = \delta_1 \left[2 - 3\frac{v^*}{U_1} I_1 + \left(\frac{v^*}{U_1}\right)^2 I_2\right] \qquad [4.5]$$

where

$$I_2 = \int_0^\infty \left[\frac{U_1 - U}{v^*}\right]^3 d\left(\frac{y\,v^*}{\delta_1 U_1}\right) \qquad [4.6]$$

is likewise practically independent of the profile for $y<\delta_w$. Inasmuch as the velocity law for y-values $\delta_w \leq y \ll \delta$ satisfies both [3.4] and [4.2], we obtain by introducing [3.4] and [4.2]

$$\frac{U_1}{v^*} = \frac{1}{\varkappa} \ln Re_1 + B \qquad [4.7]$$

where

$$Re_1 = \frac{U_1 \delta_1}{\nu} \qquad [4.8]$$

represents the Reynolds number in terms of the displacement thickness δ_1 and where

$$B = C + K \qquad [4.9]$$

The quantities I_1, I_2, and K are pure shape parameters which can be directly derived from the profile shape and which do not depend on the form of the velocity law applicable to the zone near the wall, Equation [3.3] provided condition $\delta_w \ll \delta$ is sufficiently satisfied. We shall now make the assumption that a definite correlation exists between these quantities which can be determined empirically from available measurements such as represented in Figures 4 and 5. This assumption, by the way, appears to be justified mainly on account of the slight dependence on v^*/U_1. If we approximate the velocity profile for $\delta \geq y \geq \delta_w$ by the formula

$$\frac{U_1 - U}{v^*} = \frac{A}{\varkappa}\left(1 - \frac{y}{\delta}\right) - \frac{1}{\varkappa} \ln \frac{y}{\delta} \qquad [4.10]$$

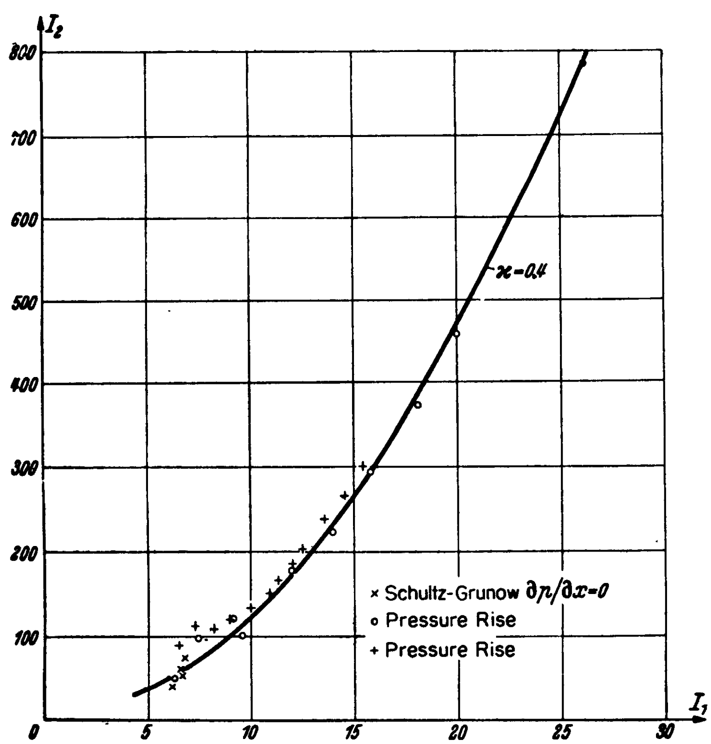


Figure 4 - Correlation between Shape Parameters I_1 and I_2 According to [4.11] and Experiments

then the relations between I_1, I_2, and B can be described approximately by the following expressions

$$\left.\begin{aligned} I_1 &= \frac{2+\frac{3}{2}A+\frac{1}{3}A^2}{\varkappa\left(1+\frac{1}{2}A\right)} \\ I_2 &= \frac{6+\frac{21}{4}A+\frac{11}{6}A^2+\frac{1}{4}A^3}{\varkappa^2\left(1+\frac{1}{2}A\right)} \\ B &= C+0.82\frac{A}{\varkappa}-\frac{1}{\varkappa}\ln\frac{1+\frac{1}{2}A}{\varkappa} \end{aligned}\right\} \qquad [4.11]$$

where A occurs as a parameter. Hence, it is possible to determine δ_2, δ_3, and c_f' if δ_1, I_1, and Re_1 and the wall roughness are given.

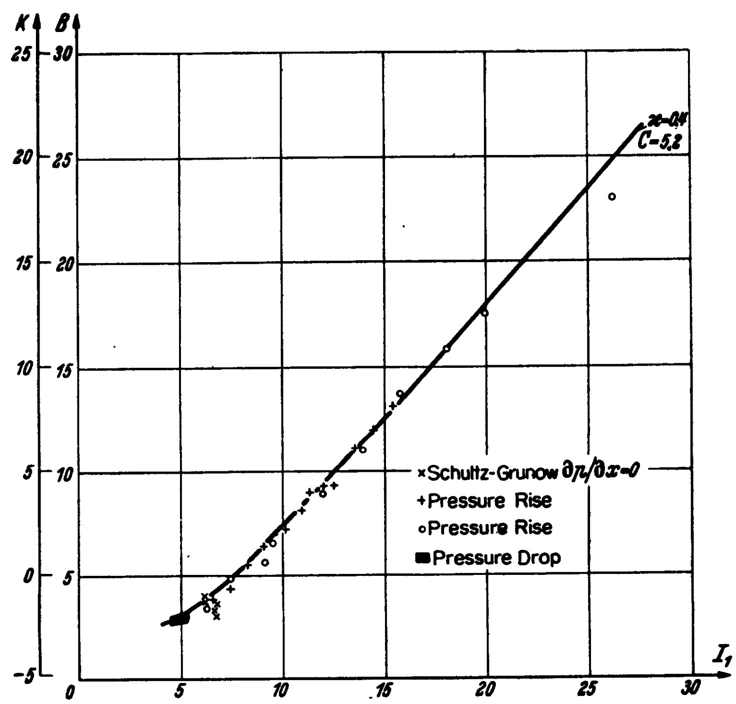


Figure 5 - Correlation between the Ratio v^*/U_1, the Reynolds Number Re_1 and Shape Parameter I_1, According to [4.11] and Experiments on Smooth Walls

5. FLUX OF TURBULENT ENERGY

In order to be able to determine the magnitude of the flux of turbulent energy by means of [2.12], it is necessary to have a 'turbulence profile,' i.e., the dimensionless plotting of the energy of turbulence E/v^{*2} against $yv^*/\delta_1 U_1$. Here, E/v^{*2} in the vicinity of the wall $y \to \delta_w$ approaches a universally applicable value and decreases very rapidly to zero for $y < \delta_w$. Although with the aid of the familiar hot-wire arrangement it is fundamentally possible to determine experimentally the root-mean-square values of all three fluctuation constants, and thus E, usable measurements from boundary layers are only available for component u which, to be sure, makes the most essential contribution to E. The longitudinal fluctuation profiles $\sqrt{\overline{u^2}}/v^*$ represented in Figures 6 and 7 were measured by W. Tillmann.[11] Inasmuch as v^*/U_1 exerts only a moderate influence on the outer velocity profile (see Section 4), we may also expect that the influence of v^*/U_1 on the turbulence profile be only very slight. Figure 6 confirms this assumption for longitudinal fluctuation profiles of the flow over a plate without pressure gradients in the case of smooth as well as rough surfaces; hence, it is reasonable to assume that the quantities related to turbulence profiles, i.e., the flux of turbulent energy and the dissipation function D defined in [2.14] can be approximately described by some unique relationship with shape parameter I_1.

As far as it was possible to ascertain up to now, the variation in the flux of turbulent energy usually has no very great effect on two-dimensional boundary-layer flows; hence, a somewhat liberal treatment of this term in [2.12] seems permissible as a rule. The order of magnitude of the v- and w-fluctuation components can be estimated from several earlier measurements in pipes and tunnels.[12] According to the available material we can assume that the relation

$$\int_0^\infty U E \, dy \approx 0.65\, \delta_1\, U_1^2\, v^* \qquad [5.1]$$

independent of I_1 is usable as an approximation.

6. DISSIPATION FUNCTION

In order to determine the dissipation function D from [2.14], we will again evaluate separately the effects of viscosity and wall roughness on the basis of the results described in Section 3. The contribution of the region near the wall $(0 \leq y \ll \delta)$ to [2.14] can easily be calculated since we may here place $\tau/\varrho = v^{*2} =$ constant, and neglect the energy of turbulence carried

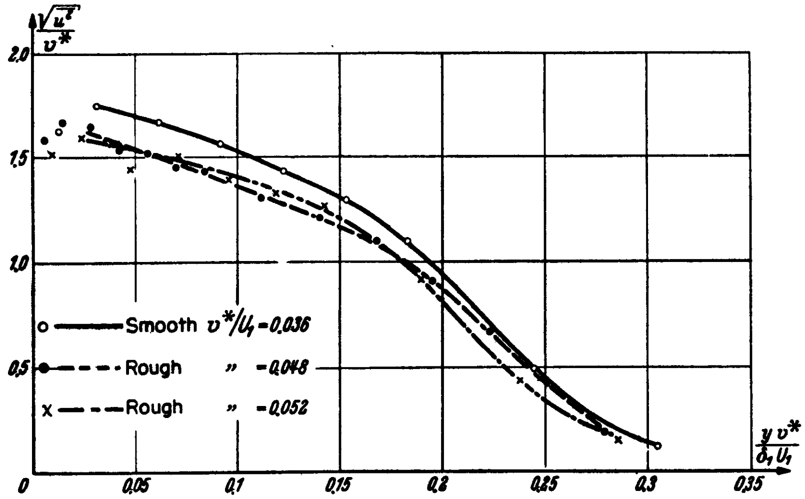


Figure 6 - Longitudinal Fluctuation Profiles at Constant Pressure, According to Measurements by W. Tillmann on Smooth and Rough Walls

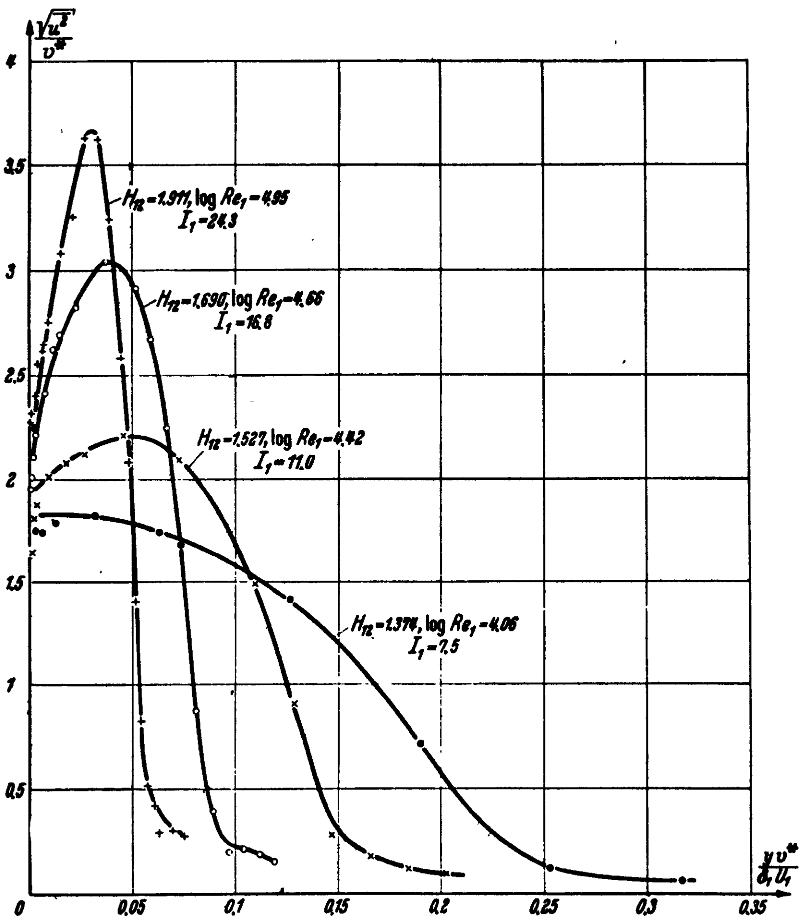


Figure 7 - Longitudinal Fluctuation Profiles in a Pressure Rise, According to Measurements by W. Tillmann

away in the mean flow. The energy dissipated in a strip of width b is then equal to the work done by the mean flow plus the kinetic energy diffusing into this strip from the outside. Thus, we obtain for $y \ll \delta$

$$\int_0^y S\,dy' = v^{*2}\,U(y) - Q(y) \qquad [6.1]$$

If the upper limit of integration y lies in the region of the universal boundary-layer flow $(\delta_w \leq y \ll \delta)$, we may introduce $U(y)$ according to [3.4]; and $Q(y)$, for reasons of flow similitude, has a value $Q = -\,c_q\,v^{*3}$ independent of y where c_q represents a universal constant. Thus we obtain for $\delta_w \leq y \ll \delta$

$$\int_0^y S\,dy' = v^{*3}\left(\frac{1}{\varkappa}\ln\frac{v^* y}{\nu} + C + c_q\right) \qquad [6.2]$$

In the case of the outer zone of the boundary layer $(y > \delta_w)$ where the Reynolds number of the turbulence is large, there applies for the dissipation [2.10] the familiar relation[2]*

$$S = c\,\frac{E^{3/2}}{l} \qquad [6.3]$$

where l represents a length designating the large elements of turbulence while c is a dimensionless factor which only depends on the structure of the turbulence. Equation [6.3] results from the fact that the kinetic energy E which is essentially contained in the largest elements of turbulence is continuously being transmitted to ever smaller elements of turbulence until it is finally transformed into heat in the smallest of these elements. In the region of the universal boundary-layer flow, l is proportional to y, for reasons of similitude, and E is constantly proportional to v^{*2}. Furthermore, the dissipation is here equal to the energy extracted from the mean flow so that for $\delta_w \leq y \ll \delta$ the following equation holds good:

$$S = c\,\frac{E^{3/2}}{l} = \frac{v^{*3}}{\varkappa y} \qquad [6.4]$$

From this, we then obtain

$$\int_y^\infty S\,dy' = \int_y^\infty c\,\frac{E^{3/2}}{l}\,dy' = v^{*3}\left(J_s - \frac{1}{\varkappa}\ln\frac{y v^*}{\delta_1 U_1}\right) \qquad [6.5]$$

*The contribution $\nu\left(\frac{\partial U}{\partial y}\right)^2$ of the mean flow is to be neglected here.

where the value of the integral expression

$$J_s = \int_{\frac{y v^*}{\delta_1 U_1}}^{\infty} c \frac{(E/v^{*2})^{3/2}}{\frac{l v^*}{\delta_1 U_1}} d\left(\frac{y' v^*}{\delta_1 U_1}\right) + \frac{1}{\varkappa} \ln \frac{y v^*}{\delta_1 U_1} \qquad [6.6]$$

is independent of the lower limit of integration y if it lies in the region $\delta_w \leq y \ll \delta$. By the addition of [6.2] and [6.5] we finally get

$$D = \int_0^{\infty} S\, dy = v^{*3}\left(\frac{1}{\varkappa} \ln Re_1 + G\right) \qquad [6.7]$$

with

$$G = C + J_s + c_q \qquad [6.8]$$

Since nothing is known regarding the behavior of the functions l and c, except in the region $\delta_w \leq y \ll \delta$, J_s cannot be calculated from [6.6] even if the turbulence profile is known. The only remaining possibility consists in calculating the function D by a differentiation wherein experimentally determined quantities are introduced into the energy equation [2.12], a method which involves a number of uncertainties. The result of the evaluation of the series of measurements carried out by F. Schultz-Grunow[9] for a flat plate without a pressure gradient (see Figure 8) confirms the correctness of the relation [6.7] regarding the effect of Reynolds number. The result of the evaluation of several series of tests by H. Ludwieg and W. Tillmann[8]* is shown in Figure 9 indicating the correlation between the dissipation function and shape parameter I_1. The scatter

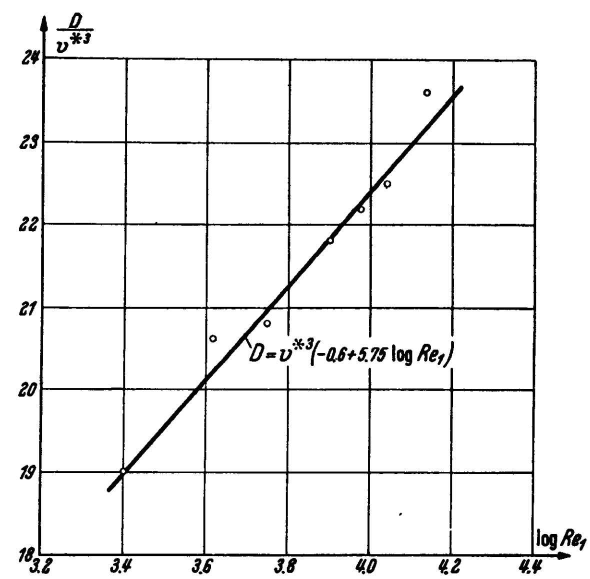


Figure 8 - Dissipation Function for the Boundary Layer without a Pressure Gradient as a Function of Reynolds Number, According to Measurements by F. Schultz-Grunow

*A number of test series involved have not been published.

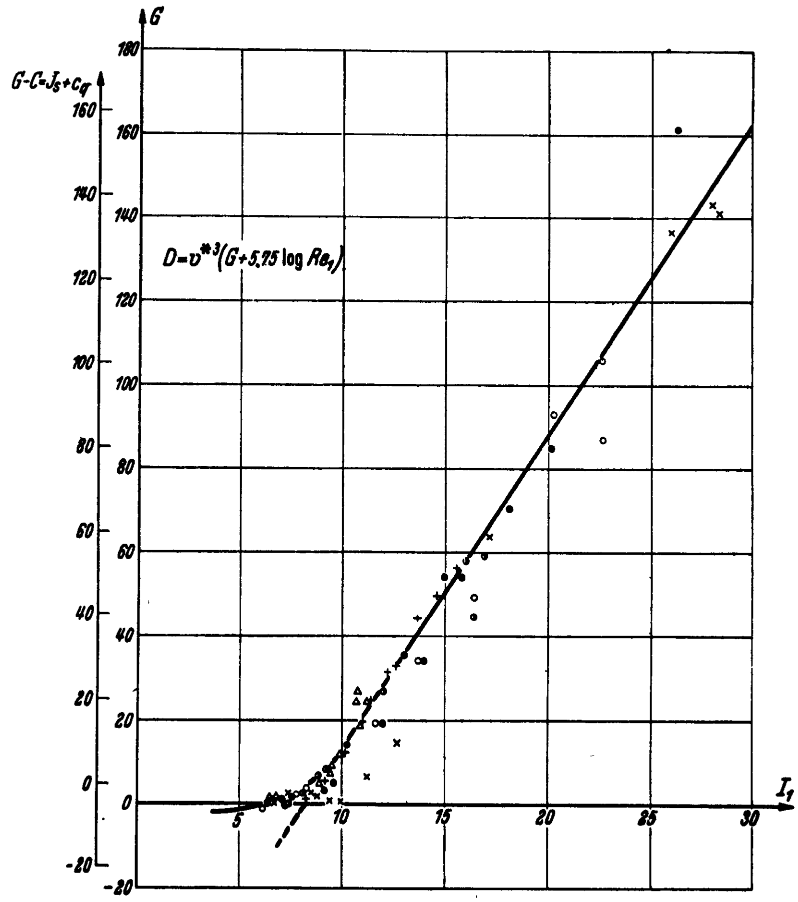


Figure 9 - Correlation Between the Dissipation Function According to [6.7], the Reynolds number Re_1 and the Shape Parameter I_1, Based on an Evaluation of Results of Measurements on Smooth Walls

of the test points is partly due to the uncertainties involved in the evaluation method. The conversion to the conditions prevailing on rough walls is possible by using [6.8] and by substituting the modified value of C from [3.4].

7. SIMILAR SOLUTIONS

Further development requires additional tests to confirm the correlations indicated as well as a theoretical analysis of the problem. By using a number of hypotheses the calculation of the turbulent boundary layer would seem to be fundamentally conceivable on the basis of boundary-layer equation [2.1] and energy-balance equation [2.9]. In that case we would have to integrate two partial non-linear differential equations instead of one as is the case for the laminar boundary layer. Also, from a mathematical standpoint we must therefore expect more difficulties with turbulent boundary layers than with laminar ones even though the latter still involve mathematical problems not to be regarded as simple by any means. At first, we shall therefore look for solutions of a simple nature on the basis of which various individual problems can be investigated. In this connection it is interesting that under certain conditions there also exist so-called similar solutions for turbulent boundary layers, i.e., solutions in which the velocity profile along the wall is only affinitatively distorted; in this case, the partial differential equations can be transformed into ordinary ones.

At first, it can be shown that for external zone ($y > \delta_w$) similar solutions exist if we neglect the viscosity in the boundary-layer equations, provided that the velocity distribution at the outer edge of the boundary layer satisfies the relation

$$U_1 = a\, x^m \qquad [7.1]$$

where a and m are constants, and that the local coefficient of friction $c_f' =$ **constant** is given. The geometrical similitude of the flow pattern requires that the boundary-layer thickness here increase linearly with x. The velocity profile can be represented as $(U_1 - U)/v^*$ over y/x and it is a function of the two parameters m and v^*/U_1. For small y/x-values it assumes the asymptotic form

$$\frac{U_1 - U}{v^*} = -\frac{1}{\varkappa}\ln\left(\frac{y}{x}\right) + K\left(m\ \frac{v^*}{U_1}\right) \qquad [7.2]$$

where the constant $K(m\ v^*/U_1)$ can be determined from the boundary-layer equations if the values of m and v^*/U_1 are given. This solution is valid only for wall distances $y \geqq \delta_w$ and must be supplemented by the velocity law [3.3] in order to obtain from the latter complete velocity profiles. The condition for the continuous transition of the outer zone to [3.3] is obtained by eliminating U/v^* in [7.2] with the aid of [3.4]:

$$\frac{U_1}{v^*} + \frac{1}{\varkappa}\ln\frac{U_1}{v^*} - K\left(m\ \frac{v^*}{U_1}\right) = \frac{1}{\varkappa}\ln\frac{U_1 x}{\nu} + C \qquad [7.3]$$

Similar solutions for the outer zone discussed here have a real significance only if the Reynolds number $U_1 x/\nu$ and the wall roughness, whose effect is expressed by the quantity C, are such that [7.3] is identically satisfied for all x-values. At very large Reynolds numbers this is the case if the length characterizing the roughness (grain size in the case of sand roughness, for instance) is proportional to x. In the case of hydrodynamically smooth walls and at constant roughness where C is a constant, the condition [7.3] cannot be rigorously satisfied for all x-values. Since x occurs logarithmically in

[7.3], we may, for sufficiently large x-values, regard the expression on the right side of [7.3] as constant, at least piece-wise, and we are justified in considering that the condition for similar solutions is thus satisfied with an accuracy sufficient for practical purposes. It is important, however—and this applies also for the developments in Sections 4 to 6—that δ_w be so small that the function $(U_1 - U)/v^*$ at the point $y = \delta_w$ does not deviate perceptibly from asymptote [7.2] or [4.2], respectively.

An evaluation of similar solutions is possible by means of the momentum equation [2.5], the energy equation [2.12] and the empirical relations in Sections 4 to 6. The results of such a calculation are shown in Figure 10 for conditions prevailing on smooth walls. It is interesting to note that a physically logical solution cannot be obtained for all m-values. This

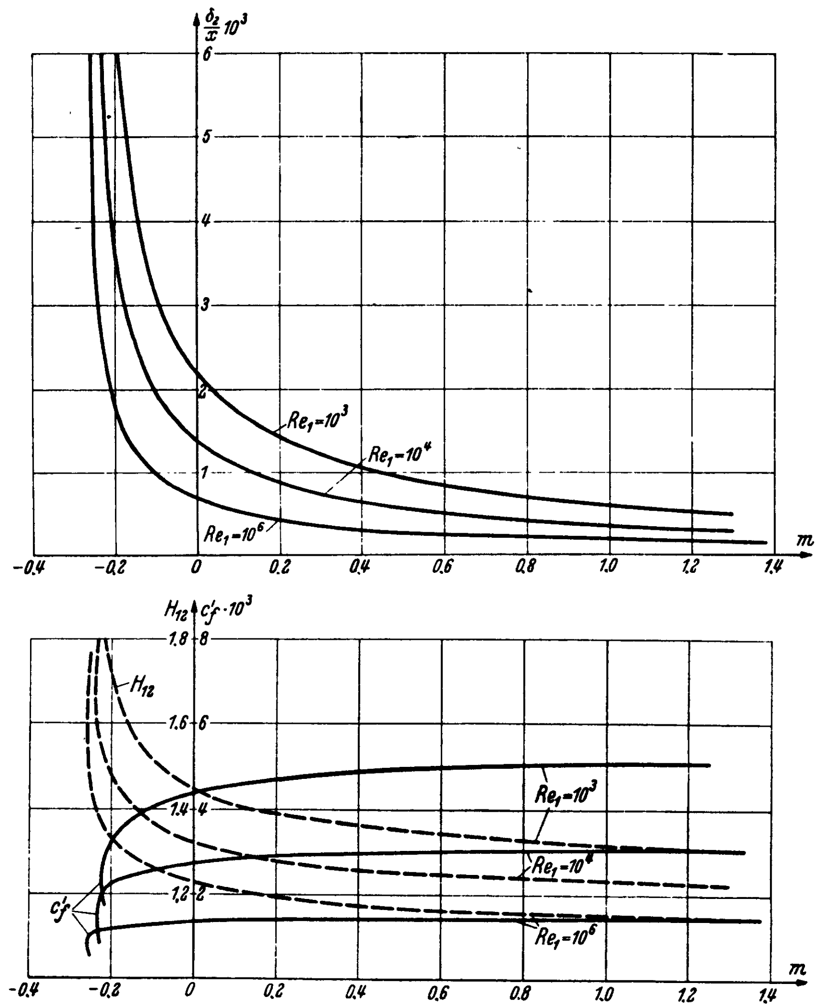


Figure 10 - Similar Solutions of the Equations for the Turbulent Boundary Layer on Smooth Walls

follows from the momentum equation [2.5] if [7.1] is introduced. Because of $U/U_1 \leq 1$ we always obtain $\delta_1 > \delta_2$ according to [2.6] and [2.7]. If U_1 is positive, we obtain negative values for δ_2 and τ_0 only if the flow breaks away from the wall. In this case, however, the conditions for boundary-layer theory are no longer fulfilled. Hence, according to the momentum equation it must be that $m > -1/3$. Figure 10 shows that in actual practice separation is to be expected approximately at $m = -0.2$. In comparison to this let us call to mind that in corresponding similar solutions for laminar boundary layers separation takes place at $m = -0.091$. This confirms the familiar empirical fact that turbulent boundary layers are able to overcome a greater pressure rise than laminar ones.

8. SUMMARY AND CONCLUSION

It is advantageous to separate the turbulent boundary layer into two regions (see Figure 1) which are joined by a common intermediate zone $(\delta_w \leq y \ll \delta)$. In the outer zone $(y \geq \delta_u)$ the velocity profile is essentially a function of the velocity distribution U_1 along the outer edge of the boundary layer only; it can be approximately represented as a function which is proportional to the friction velocity $v^* = \sqrt{\tau_0/\varrho}$; otherwise, however, it is independent of viscosity and the processes on the wall. In the region near the wall $(0 \leq y \ll \delta)$ the velocity profile depends on viscosity and wall roughness, but is only indirectly, i.e., through parameter v^*, a function of the velocity distribution of the outer flow. In the intermediate zone a universal law of flow applies which is a function of v^* only while remaining unaffected by wall roughness and viscosity as well as by the outer flow.

If we take this into consideration, the evaluation of test data furnishes empirical relations between the various quantities which, together with the indicated integral theorems for the balance of momentum and of energy, may be used for an approximate calculation of turbulent boundary layers. If we compare a method of calculation developed on this basis with the methods made known thus far, we find that the former method offers the advantage that the coefficient of friction can be determined more reliably as a function of the Reynolds number and the given velocity distribution, apart from the fact that this method can also be applied to rough walls. The velocity profiles are two-parametric in this case since, in accordance with the existing Reynolds number a velocity law [3.3] is adapted to the outer zone [4.1] which is assumed to have but one parameter. When actually carrying out boundary-layer calculations, it is necessary to reduce the indicated relationships to a more convenient form, a procedure which can be accomplished without difficulty.

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