Translation 337

SIMPLIFIED ANALYSIS OF THE NATURAL FREQUENCIES FOR BENDING VIBRATIONS OF SHAFTS

NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
WASHINGTON, D.C. 20007

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(Uproshchennyi Raschet Sobstvennykh Chastot Izgibnykh Kolebanii Valov)

by

M.S. Movnin and E.L. Akselrad
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ABSTRACT*

Mathematical techniques based on the principles of minimum potential energy and minimum complementary energy are discussed. Two numerical examples of practical interest, dealing with the vibration of nonuniform whirling shafts, are worked out to illustrate the closer approximations for frequency resulting from the method based on minimum complementary energy.

SUMMARY**

The Rayleigh quotient plays an important role in the theory of eigenvalue problems arising in mathematical physics. In the particular application of finding approximations to the natural frequencies for linear, free bending vibrations of slender, nonuniform elastic beams, the Rayleigh quotient (as it derives from the "principle of minimum potential energy") has the form

$$\tilde{\omega}^2 = \min.\left\{ \frac{\int_0^L EI(x)\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L \rho(x)\; y^2(x)\,dx} \right\} \geq \omega^2_{exact} \qquad \text{(a)}$$

where EI(x) and $\rho(x)$ are the bending stiffness and mass distributions, respectively, along the length L of the beam; y(x) is the transverse displacement due to the vibratory motion.

An important property of the quotient (a) is a theorem, due to Rayleigh himself, which states that the lowest (fundamental) eigenfrequency is given by the minimum of the quotient for all admissible functions y(x) which satisfy the geometric boundary conditions and are well-behaved in their continuity and differentiability. For the true eigenfunctions (natural modes) of the beam, the Rayleigh quotient yields the corresponding exact eigenfrequencies (natural frequencies).

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*There was no abstract in the original paper by Movnin and Akselrad; this abstract was prepared by the translators.

**This summary was prepared by the translators.

In the absence of knowing the true modes, substitution of a trial function consisting of a single term with one arbitrary multiplicative coefficient into Equation (a) will yield a satisfactory approximation $\tilde{\omega}$ provided the chosen function closely represents the actual configuration of the beam, and also if the second derivative is a close representation of the beam curvature. An important consequence of Rayleigh's theorem is the principle that the quotient (a) gives an approximation $\tilde{\omega}$ possessing a second-order error if the trial function substituted therein deviates by first-order error from a natural mode.

However, since successive differentiation of an approximation function perpetuates the inherent error so that although, in practice, it is possible to choose trial functions which closely resemble the natural modes, it is not to be expected that they will also satisfy the condition

$$\left[y''^{\,2}(x)\right]_{\text{approx.}} \approx \left[y''^{\,2}(x)\right]_{\text{exact}} \qquad \text{(b)}$$

The difficulty of satisfying (b) increases the inaccuracy of the Rayleigh quotient (a).

To increase the accuracy of the frequencies thus computed, it is possible to consider approximations to y(x) comprised of a linear combination of several linearly-independent functions from the set of admissibles; this is the essence of Ritz's generalization of the Rayleigh method. Although such a procedure increases the numerical calculations necessary to improve the accuracy of the lowest frequency $\tilde{\omega}_1$, it does yield approximations to several of the higher harmonics depending on the number of coordinate functions spanning the Ritz manifold. An alternative to this so-called Rayleigh-Ritz procedure was first proposed by Grammel (see Reference 2)* in which a one-term representation of y(x) may be taken as before, but instead of using Equation (a) to determine approximate values of the frequencies, a modified form of the Rayleigh quotient is

*The original paper by Grammel has recently been translated from the German by the present translators and is available as David Taylor Model Basin Translation 335 (Oct 1966).

employed. The details of the pertinent derivation will be indicated for the relatively simple case of a cantilever beam, say, to delineate the salient features of the modified Rayleigh quotient.

If we assume that the beam is vibrating in one of its principal (natural) modes, the differential equation governing the free bending motions is given by

$$\left(EI(x)y''\right)'' = \omega^2\rho(x)y \tag{c}$$

Two successive integrations of this equation from one end of the beam, x = 0 say, to some arbitrary station x along its length yield

$$EI(x)y'' = \omega^2 \int_0^x \int_0^\eta \rho(\xi)\ y(\xi)d\xi\ d\eta + \Big[(EIy'')'\Big]_{x=0}^{x} + \Big[EIy''\Big]_{x=0} \tag{d}$$

Thus, if the beam is free of shearing force and bending moment at its end x=0, i.e., $\Big[(EIy'')'\Big]_{x=0} = 0 = \Big[EIy''\Big]_{x=0}$, Equation (d) simplifies to

$$y''(x) = \frac{\omega^2}{EI(x)} \int_0^x \int_0^\eta \rho(\xi)\ y(\xi)d\xi\ d\eta \tag{e}$$

If the expression (e) is substituted for the curvature $\frac{d^2y}{dx^2}$ in the numerator integrand of the Rayleigh quotient (a), the following is obtained:

$$\tilde{\tilde{\omega}}^2 = \min. \left\{ \frac{\int_0^L \rho(x)\ y^2(x)dx}{\int_0^L \frac{1}{EI(x)}\left[\int_0^x \int_0^\eta \rho(\xi)y(\xi)d\xi d\eta\right]^2 dx} \right\} \geq \omega^2_{exact} \tag{f}$$

Equation (f) is sometimes referred to as the modified Rayleigh quotient as it derives from the "principle of minimum complementary energy."

The basic difference between the energy quotients (a) and (f) stems from the manner in which the potential (strain) energy is expressed in each. Implicit in finding approximations $\tilde{\omega}$ using the quotient (a) is the fact that the governing differential equation (c) is satisfied approximately, but the stress-strain relation is satisfied exactly since the expression, $M(x) = -EI(x)y''$, is used explicitly to determine the bending moment by differentiation. Implicit in finding approximations $\tilde{\tilde{\omega}}$ using the quotient (f) is the fact that the equilibrium equation is satisfied exactly since it is integrated directly to determine the bending moment, whereas the stress-strain relation is satisfied approximately. The nature of the approximation is thus seen to be different in the two energy quotients (a) and (f).

It has been shown, from the results of computations stemming from several independent investigations,* that the Rayleigh and Rayleigh-Ritz methods based on the energy quotient (f) yield approximations $\tilde{\tilde{\omega}}$ which are closer to the exact natural frequencies ω than the approximations $\tilde{\omega}$ based on the energy quotient (a). Movnin and Akselrad give two examples of application of the modified Rayleigh quotient (f) which further establishes the validity of this important result.

*One such investigation was carried out in 1947-48 by J.G. Pulos under the direction of Prof. E. Reissner at the Mass. Institute of Technology; it culminated in an M.S. dissertation entitled, "Coupled Flexural-Torsional Vibrations of Elastic Beams."

Simplified Analysis of the Natural Frequencies for Bending Vibrations of Shafts

by M.S. Movnin and E.L. Akselrad

(Translated from the Russian by B.V. Nakonechny and J.G. Pulos, DTMB)

There are several methods which deal with the calculation of natural frequencies of vibration. A summary of such methods is given in Reference 1.*

When classifying problems of calculating the natural frequencies of vibration, with regard to those being the most complicated and which require the application of the Rayleigh-Ritz method, one considers the problems of determining the effect of varying rigidity distribution along the length with the need for including a varying mass distribution or more than two concentrated masses.[1] As a matter of fact, to this class of problems belongs the calculation of the natural frequencies of high-speed shafts and spindles which are characterized by large self-weight as compared to the weight of components mounted on them. For structural parts of the hanging-support type the inclusion of the mass distribution is always necessary, because in the majority of such cases the distributions of mass and rigidity are highly nonuniform.

Thus, approximate methods are very useful in calculating the vibration characteristics of the principal components of machine elements.

Application of the Rayleigh-Ritz method is limited by the fact that even in simple cases it requires a large amount of involved computations as compared to some other methods, and in complex cases where the use of approximate methods is required, it is necessary to have special experience and a great deal of time for the computations.

One can achieve a substantial simplification in the calculation of natural frequencies using the Rayleigh method by performing these calculations in general form for basic types of mechanisms.

The analysis of structures shows that calculation of the transverse vibrations of spindles and shafts of machines is possible to perform since the majority of cases fall in the category of beams on two supports as

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*References are listed on page 18.

shown in Figure 1, where: $q(x)$ is the weight per unit length of the beam, $J(x)$ is the bending moment of inertia of the beam along its length, and $J(o)$ is the value of $J(x)$ at a reference section.

The most critical transverse vibrations occur in such machine tool parts as cantilevers and cross-beams carrying working heads. The computation scheme for these cases is shown in Figure 2.

Below are shown the simple formulas for direct calculation of natural frequencies for the typical cases shown in Figures 1 and 2.

For cases which do not fall in the category of the typical ones shown in these figures, it is recommended that a modified Rayleigh method be used which substantially reduces the complexity of the calculations.

The starting point in calculating the natural frequencies by the Rayleigh method is the equating of the maximum value of the kinetic energy (T_m) of the vibrating masses to the potential energy (Π_m) of the deformed system. For the case of "free" vibrations, and in the absence of damping, these quantities separately are equal to the total energy of the system, i.e.

$$T_m = \Pi_m \qquad [1]$$

The appropriate form for this equality can be written as:

$$\frac{1}{2}\int_L \frac{q(x)}{g}\left(\frac{dy}{dt}\right)^2_{max} dx = \frac{1}{2}\int_L \frac{M_m^2(x)}{EJ(x)}\,dx \qquad [2]$$

$$M_m = EJ\,\frac{d^2y_m}{dx^2}$$

where $M_m(x)$ is the maximum bending moment at the section, and $\frac{q(x)}{g} \equiv \rho(x)$.

For small bending vibrations the deflection of the beam, which vibrates in a principal mode,* is given by the expression

$$y(x,t) = \theta(x)\,\sin\omega t;\ \left(\frac{dy}{dt}\right)_{max} = \omega\theta(x) \qquad [3]$$

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*The principal mode of free vibration is that for which all points of the system vibrate with the natural frequency ω.

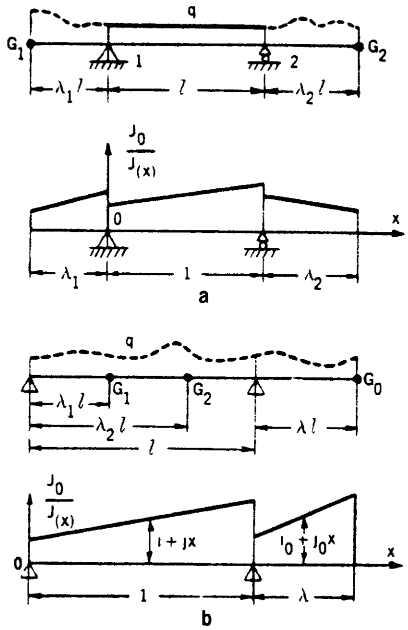


Figure 1

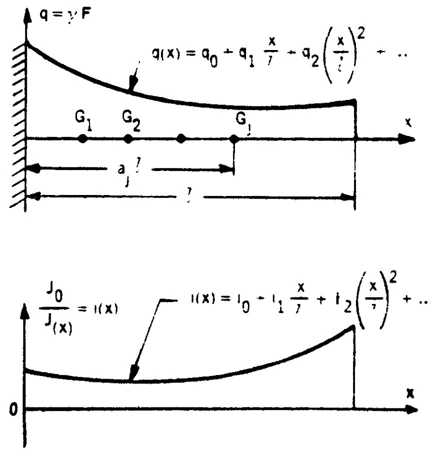


Figure 2

Using Equations [2] and [3] the Rayleigh quotient, which relates the natural frequency ω with the principal mode of vibration θ(x), is found to be,

$$\omega^2 = \frac{\int_L EJ(\theta'')^2 dx}{\int_L \rho(x)\theta^2 dx} \qquad [4]$$

If in this formula we substitute, in place of the function θ(x), some other function $\boldsymbol{\theta}(x)$,* we obtain the following value, which according to Rayleigh's principle will be larger than the natural frequency:

$$\tilde{\omega}^2 = \frac{\int_L EJ(\boldsymbol{\theta}'')^2 dx}{\int_L \rho(x)\boldsymbol{\theta}^2 dx} > \omega^2 \qquad [5]$$

*Translator's Note: Strictly speaking, the functions $\boldsymbol{\theta}(x)$ must be of the (essentially) admissible type in the sense that they, at least, satisfy the geometric boundary conditions of the beam.

In order that $\tilde{\omega}$ be as close as possible to the natural frequency ω, the function $\boldsymbol{\theta}(x)$ should accurately represent the form of the deflection of the beam, corresponding to the principal mode of vibration $\theta(x)$, i.e., $\boldsymbol{\theta}(x) \simeq \theta(x)$, and the squares of their second derivatives should be approximately equal, i.e., $(\boldsymbol{\theta}'')^2 \approx (\theta'')^2$. The difficulty in choosing the function $\boldsymbol{\theta}(x)$ so as to satisfy these conditions increases the error inherent in Equation [5].

In the theory of Rayleigh's principle the method of successive approximations is proven. In accordance with this method of successive approximations, Equation [5] gives the zero-th approximation to ω.

Consequently, the first approximation can be obtained by replacing the function $\boldsymbol{\theta}(x)$ in Equation [5] by the function $\boldsymbol{\theta}_1(x)$ which is equal to the deflection of the beam under the action of the distributed loading

$$q_{\boldsymbol{\theta}} = \rho(x)\boldsymbol{\theta}(x)$$

Equating the energy of deformation of the beam to the work of the loading which caused this deformation, i.e., $\int_L EJ(\boldsymbol{\theta}_1'')^2 dx = \int_L q_{\boldsymbol{\theta}}\boldsymbol{\theta}_1 dx$, we can then write the equation for the first approximation in the following form:

$$\tilde{\omega}^2 = \frac{\int_L q_{\boldsymbol{\theta}}\boldsymbol{\theta}_1(x)\,dx}{\int_L \rho\boldsymbol{\theta}_1^2(x)\,dx} = \frac{\int_L \rho\boldsymbol{\theta}\boldsymbol{\theta}_1 dx}{\int_L \rho\boldsymbol{\theta}_1^2 dx} \qquad [6]$$

In a similar manner we can develop the second approximation, after having found the deflection $\boldsymbol{\theta}_2(x)$ from the loading which is given by $q'_{\boldsymbol{\theta}} = \rho(x)\,\boldsymbol{\theta}_1(x)$, ... etc.

The Rayleigh method is especially applicable to the calculation of shafts having variable rigidity, for which the representation of the elastic line (the determination of $\boldsymbol{\theta}_1$) requires considerable graphical work.

Hence, in many cases one is restricted to the zero-th approximation even though Equation [5] doesn't give the desired accuracy.

As was already mentioned, the inaccuracy of Equation [5] is caused by the presence in the numerator of the square of the second derivative of the arbitrarily chosen function $\theta(x)$, and by the difficulty in calculating the first approximation from Equation [6]—due to the necessity of calculating the deflection of the beam under the complex loading $q_\theta = \rho(x)\ \theta(x)$.

Let us derive a formula which does not contain the second derivative of the arbitrarily chosen function and which doesn't require the calculation of the deflection due to the loading q_θ.

For this purpose let us turn to the expression which deals with the potential energy due to bending deformations. If the beam is vibrating in a mode given by the function

$$y(x,t) = \theta(x) \sin \omega t \qquad [7]$$

then the motion of the beam due to the action of the inertia loading

$$\rho(x) \frac{d^2 y}{dt^2} = - \omega^2 \rho(x)\ \theta(x) \sin \omega t \qquad [8]$$

can be determined.

Let us denote the bending moment at a cross-section by $M_\theta(x)$, which is due to the transverse loading $q_\theta = \rho(x)\ \theta(x)$. It is evident, that the maximum value of the bending moment at the section is calculated from the equation

$$M_m(x) = \omega^2\ M_\theta(x) \qquad [9]$$

Substituting the expressions given by Equations [7] and [9] into Equation [2], we get the following formula for determining the approximate value of the natural frequency:[2,3]

$$\tilde{\tilde{\omega}}^2 = \frac{\int_L \rho(x)\ \theta^2(x) dx}{\int_L \frac{1}{EJ(x)}\ M_\theta^2(x) dx} \qquad [10]$$

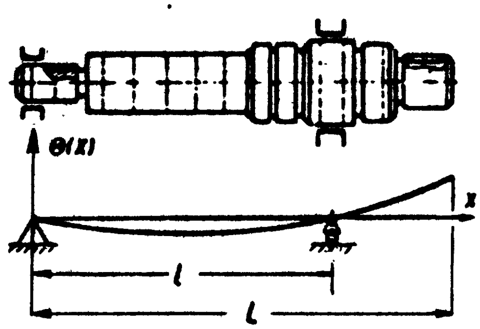
For calculation purposes it is convenient to modify this last expression by introducing nondimensional quantities and by separating the concentrated masses, i.e.,

$$\tilde{\tilde{\omega}}^2 = \nu^2 \left[\int_L \frac{\rho(x)}{\rho_o} \theta^2(\xi) d\xi + \frac{m_i}{\rho_o \ell} \theta^2(\xi_i) \right] \left[\int_L \frac{J_o}{J(x)} \mu_\theta^2(\xi) d\xi \right]^{-1} = \nu^2 \frac{\beta}{\alpha} \quad [11]^*$$

where $\nu^2 = \frac{EJ_o}{\rho_o \ell^4}$; ℓ is the length between supports or the length of the cantilever (see Figures 1 and 2); $\xi = x/\ell$ is the nondimensional coordinate for a transverse cross section of the beam or shaft; $\mu_\theta(\xi)$ is the nondimensional bending moment of the beam in terms of the longitudinal coordinate $\xi = \frac{x}{\ell} = 1$, due to the nondimensional loading $\frac{\rho(\xi)}{\rho_o}\theta(\xi) + \frac{m_i}{\rho_o \ell}\theta(\xi_i)$; ρ_o and J_o are the values of the mass per unit length of the shaft and moment of inertia at some reference cross section, respectively; m_i, ξ_i denote the i^{th} mass and its corresponding coordinate; and for a shaft with $\rho(x) \approx 0$, we take $\rho_o \ell = m_o$.

Example 1. Using Equation [11], let us calculate the natural frequency of vibration for a whirling shaft (Figure 3); this problem was considered in Reference 1.

Figure 3



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*Translators' Note:
The variable of integration is the nondimensional quantity $\xi = \frac{x}{\ell}$ so that $\frac{\rho(x)}{\rho_o} \rightarrow \rho(\xi)$ and $\frac{J_o}{J(x)} \rightarrow \frac{1}{J(\xi)}$ in the equation. Also, the limits on the integrals of Equation [11] over the length of the beam should be 0 to 1.

We determine the numerator and denominator of Equation [11], using intermediate results from Reference 1, by taking as in Reference 1,

$$\theta(x) = \sin\frac{\pi x}{L} - \frac{x}{\ell}\sin\frac{\pi x}{L}$$

From Reference 1,

$$\int_L \rho(x)\theta^2 dx = \frac{24.95}{g}\ \text{kg-cm}^2$$

The denominator of Equation [11] can be found by numerical integration and using the values of $S_i = (M_{i-1} + M_i)\frac{\Delta x_i}{2}$ and $\frac{J_o}{J(x_i)}S_i$ given in Table 23 of Reference 1; i.e.,

$$\int_L \frac{1}{EJ(x)} M_\theta^2(x)\,dx = \frac{1}{EJ_o g^2}\sum_i S_i \frac{J_o}{J(x_i)}\frac{S_i}{\Delta x_i}$$

After substitution and summation

$$\int_L \frac{1}{EJ(x)} M_\theta^2(x)\,dx = \frac{0.565\text{x}10^{-2}}{g^2}\ \text{kg-cm}$$

Putting these values into Equation [10] we get

$$\tilde{\tilde{\omega}} = 2040\ \text{sec}^{-1}$$

This approximate value for the frequency is more accurate than that obtained in Reference 1, e.g., $\tilde{\omega} = 2125\ \text{sec}^{-1}$; the difference between these two values can be explained as due to errors arising from the graphic integration used in Reference 1.

At the same time, the amount of numerical work involved in using Equation [11] is much less than that required when using Equation [6], since in this case there is no need to calculate the deflection of the beam under the action of a complex loading distribution.

Let us develop the formulas necessary for calculating the natural frequencies of shafts with variable rigidity distribution along their length and with cantilevered sections as shown in Figure 1a. In this computation a linear variation of the quantity $J_o/J(x)$, which is the inverse of the moment of inertia of the shaft cross-section in bending, is assumed. Also a linear distribution for the mass over the span, and an arbitrary distribution for the overhang are assumed.

To include the masses which are on the overhang portions of the beam, we can reduce them to a single mass by using the formula of Reference 1, which, with the notation of Figure 4, can be written as follows:

$$m = m' + m'' \left(\frac{\lambda''}{\lambda'}\right)^2 \left(\frac{1 + \lambda'}{1 + \lambda''}\right) \quad [12]$$

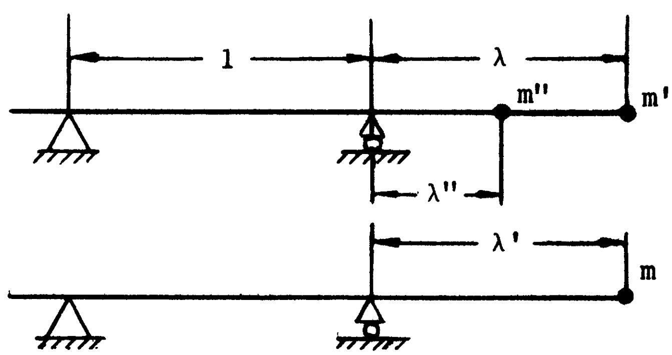

Figure 4

The compliance to deformation for each portion of the shaft (see Figure 1a) can be approximated by a linear relationship, i.e.

$$\frac{J_o}{J(x)} = \begin{cases} i + jx, & \text{for the length } \ell \\ i_1 + j_1 x, & \text{for the length } \ell_1 (= \lambda_1 \ell) \\ i_2 + j_2 x, & \text{for the length } \ell_2 (= \lambda_2 \ell) \end{cases}$$

Now we assume the following form for the elastic line:

$$\theta(x) = 3(x^2 - x)$$

After calculation using Equation [11], we get

$$\beta = m_1 \lambda_1^2 (3 + 2\lambda_1)^2 + m_2 \lambda_2^2 (3 + 2\lambda_2)^2 + 0.30 \quad [13]$$

$$\alpha = \mu_2^2\left(\frac{\lambda_2}{3} i_2 + \frac{\lambda_2}{4} j_2 + \frac{i}{3} + \frac{j}{4}\right) +$$

$$+ \mu_1^2\left(\frac{\lambda_1}{3} i_1 + \frac{\lambda_1}{4} j_1 + \frac{i}{3} + \frac{j}{12}\right) + \mu_1 \mu_2\left(\frac{i}{3} + \frac{j}{6}\right) +$$

$$+ \left(\mu_1 + \mu_2\right)\frac{i}{20} + \left(1.5 \mu_2 + \mu_1\right)\frac{j}{50} + 0.031 (i + 0.9j) \qquad [14]$$

where

$$m_i = \frac{G_i}{q\ell} ; \mu_1 = m_1 \lambda_1^2 (3 + 2\lambda_1); \mu_2 = m_2 \lambda_2^2 (3 + 2\lambda_2)$$

Now we will consider an example which demonstrates the reduction in work which results by using Equation [11] as compared to the calculations with the Rayleigh method.

Example 2. Using Equations [11], [13], and [14], let us calculate the first natural frequency for a whirling shaft (see Figure 3).

As can be seen from Figure 5a, the distribution of the moment of inertia along the length of the shaft is variable, which fact substantially complicates the analysis. As was already mentioned, a quite complicated analysis based on graphical methods for determining the elastic line $\dot{H}_1(x)$ is given in Reference 1. Calculations using Equation [11] are considerably simpler; however, even here a substantial amount of work is required in calculating the quantities α and β.

At first glance the plots of Figure 5 show that Equations [12] to [14] may not be applicable since the function $J_o/J(x)$ is not linear and the weight of the shaft per unit length $q = \gamma F$ is not constant along the length.

However, a closer examination shows that it is possible to apply these equations. Let us construct on Figure 5b a set of straight lines $i^+ + xj^+$ and $i^- + xj^-$, representing upper and lower bounds, respectively, to the bending-compliance characteristics of the shaft, and then determine the corresponding values of the parameters i^-, j^-, i^+, j^+.

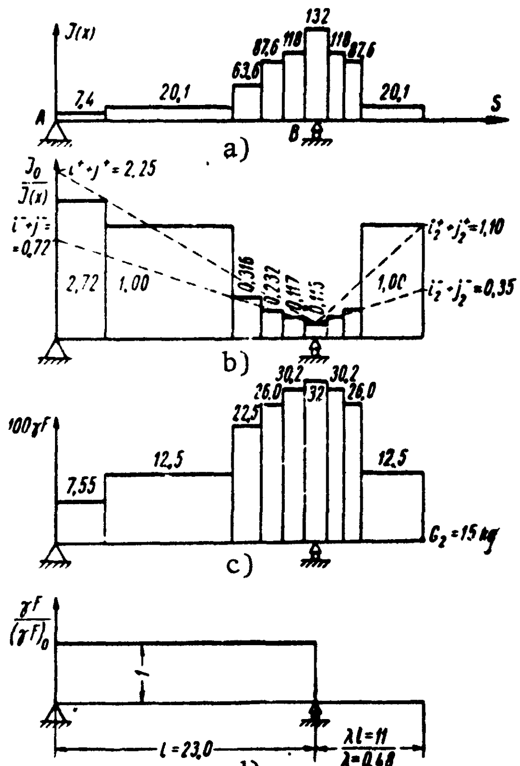

Figure 5

The mass distribution of the shaft can be considered to be uniform. (Note: a portion of the mass which is located near the support has a small influence on the vibration characteristics.)

We now obtain the parameters for the approximation to the shaft rigidity (Figure 5b). For the lower-bound distribution of compliance we get, $i^- = 0.72$; $j^- = -(0.72-0.15) = -0.57$; $i_2^- + j_2^- = 0.35$; $j_2^- = 0.35-0.15 = 0.20$; and, for the upper-bound distribution we get, $i^+ = 2.25$; $j^+ = -(2.25-0.15) = -2.10$; $i_2^+ = 1.10$; $j_2^+ = 1.10 - 0.15 = 0.95$.

We now determine the mass of the shaft. The actual distribution is shown in Figure 5c. The distribution used in the analysis (Figure 5d) is divided up into a uniform loading $(\gamma F)_0 = 0.125$ kg/cm between the supports, and a loading which is taken into account by the concentrated mass

$$G_1 = 7.5\left(\frac{22.2 + 32}{2} - 12.5\right) 0.01 \approx 1.10 \text{ kg}$$

The relative magnitude of this concentrated loading is

$$m_{1AB} = \frac{G_1}{(YF)_o \ell} = 0.374$$

The loading on the shaft overhang is reduced, by using Equation [12], to a single mass m_2 at the cross-section where the concentrated weight $G_2 = 15$ kg:

$$m_2 = 0.374 \left[15 + (0.302 + 0.26)2 \cdot \frac{3^2(23+3)}{11^2(23+11)} + (0.125)6 \cdot \frac{8^2(23+8)}{11^2(23+11)} \right] = 5.28$$

Evidently, one does not need to include m_{1AB} since $m_{1AB} << m_2$.

Let us calculate the natural frequency corresponding to the upper-bound rigidity distribution (i^-, j^-). After substituting into Equations [13] and [14], $m_1 = 0$; $\lambda_1 = 0$; $\lambda_2 = 0.48$, we obtain $\mu_1 = 0$; $\mu_2 = 4.83$; $\beta = 19.40$; $\alpha = 3.15$ and the upper-bound value of the natural frequency is found to be

$$\omega^+ = \sqrt{\nu^2 \frac{\beta}{\alpha}} = 2720 \text{ sec}^{-1}$$

Let us calculate the natural frequency corresponding to the lower-bound rigidity distribution. After substituting in Equation [14] the values of i^+, j^+, we obtain $\alpha = 6.95$, $\omega^- = 1760$ sec.$^{-1}$

The exact value of the natural frequency is close to the arithmetic mean of ω^+ and ω^-, i.e., $\omega_1 \approx 2240$ sec.$^{-1}$

After extensive calculations in Reference 1, the value $\omega = 2125$ sec^{-1} was obtained. The difference with our result is 5.4 percent, i.e., the approximate analysis gives quite satisfactory accuracy.

For the case of the shaft shown in Figure 1b, using the deflection function $\theta(x) = 3\,(x^2 - x)$, we obtain from Equation [11]

$$\beta = m_1 f_1^2 + m_2 f_2^2 + m_o f_o^2$$

$$\alpha = y_1\left(\frac{3\lambda_2 - 1}{6}\right) + y_2\left(\frac{1-\lambda_1}{2}\right) + y_3\left(\frac{1-\lambda_2}{2}\right) + y_{30}\,\frac{\lambda}{3}$$

In these expressions

$$m_1 = \frac{G_1}{q_o \ell} + \Delta_1 \;;\; m_2 = \frac{G_2}{q_o \ell} + \Delta_2$$

$$f_1 = 3\left(\lambda_1^2 - \lambda_1\right) \;;\; f_2 = 3\left(\lambda_2^2 - \lambda_2\right) \;;\; f_o = \lambda(3+3\lambda)$$

$$y_1 = (\mu_1 + \lambda_1\mu_2 + \lambda_1\mu_o)^2\ (i + \lambda_1 j)$$

$$y_2 = \left[\mu_1\left(\frac{1-\lambda_2}{1-\lambda_1}\right) + \mu_2 + \lambda_2\mu_o\right]^2 (i + \lambda_2 j)$$

$$y_3 = \mu_o^2(i + j);\; y_{30} = \mu_o^2(i_o + j_o);\; \mu_1 = m_1 f_1 \lambda_1(1-\lambda_1)$$

$$\mu_2 = m_2 f_2 \lambda_2(1-\lambda_2);\; \mu_o = m_o f_o \lambda$$

where m_i is the weight concentrated at the points i = 1, 2 divided by the weight of the shaft between the supports (ql); Δ_1, Δ_2 are the portions of the shaft weight reduced to the values G_1, G_2. If $m_1+m_2 \leq 1$, i.e., the concentrated weight of the span is smaller than the weight of the shaft, then it is imperative to use the computation scheme which we considered earlier (see Figure 1a), including concentrated masses by increasing the mass of the shaft.

The reduced mass of the overhang resulting from Equation [12], located at the distance $\lambda\ell$ from the support, is $m_o = G_o/q\ell$.

The formulas given above could be generalized to include a large number of masses G_1, G_2, G_3,...etc.

The natural frequency of the cantilever beam shown in Figure 2 is determined in an analogous fashion. Taking the deflection function to be $\theta(x) = x^2$, we obtain approximately

$$\beta = \sum_n \rho_n\left(\frac{1}{n+5}\right) + \sum_j m_j a_j^4 \;;\; \rho_n = \frac{q_n}{q_o} \;;\; m_j = \frac{G_j}{q_o \ell}$$

$$\alpha = \sum_n i_n \left\{\sum_j k_{jn}\, m_j a_j^2 + \sum_k \gamma_{kn}\, \rho_k\right\}^2$$

The coefficients k_{jn}, γ_{kn} are determined from the following table:

Coeff.	$n=0$	$n=1$	$n=2$	$n=3$
k_{jn}	$0{,}55\, a_j\sqrt{a_j}$	$0{,}29\, a_j^2$	$0{,}18\, a_j^2\sqrt{a_j}$	$0{,}13\, a_j^3$
γ_{0n}	0,224	0,091	0,0487	0,03
γ_{1n}	0,162	0,07	0,0387	0,025
γ_{2n}	0,127	0,056	0,032	0,022
γ_{3n}	0,104	0,046	0,027	0,018

The calculation of the natural frequency for a milling-cutter shaft possessing the unusual distribution of mass and rigidity considered in Reference 1 is here found to be $\tilde{\tilde{\omega}} = 300\ \text{sec}^{-1}$. This result, obtained by using $i = 0.14 + 1.16x^2$; $\rho_0 = 20$; $\rho_1 = -20$; $m_1 = 2.68$; $m_2 = 1.5$; $a_1 = 0.82$; $a_2 = 1$, is 22.5 percent smaller than the value $\omega = 387\ \text{sec}^{-1}$ given in Reference 1.

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13. ABSTRACT

Mathematical techniques based on the principles of minimum potential energy and minimum complementary energy are discussed. Two numerical examples of practical interest, dealing with the vibration of nonuniform whirling shafts, are worked out to illustrate the closer approximations for frequency resulting from the method based on minimum complementary energy.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Eigenvalues--Mathematical Analysis						
Beams--Vibration						
Whirling Shafts, Vibration						
Rayleigh-Ritz Method						
Lower and Upper Bound Solutions						
Rayleigh Method						

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